People's Democratic Republic of Algeria

Ministry of Higher Education and

Scientific Research

University of Tissemsilt

Title:

Physics of Vibrations

(with problems corrected)

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Foreword

This handout is intended for 2 LMD students with the following specialties:

- **Technological Sciences (ST):** Electrotechnics (ETT), Electronics (EN), Mechanical Engineering (GM), Hydraulics (Hyd) and Civil Engineering (GC).
- **Material Sciences:** common bases, Physic, chemistry.

This document is a detailed course. It includes five chapters cited below:

Chapter 1: General information on oscillations.

Chapter 2: Free oscillations with one degree of freedom.

Chapter 3: Oscillations damped to one degree of freedom.

Chapter 4: Oscillations forced to one degree of freedom.

Chapter 5: Free oscillations of systems with several degrees of freedom

The objective is to give students elements which will allow them to enrich their knowledge on the one hand and on the other hand help them to better master the problems they may encounter in this module.

Contents

:

Chapter 1:

General information on oscillations

1. Definitions :

1.1. The vibration :

Is an oscillatory physical phenomenon of a body moving around its equilibrium position.

1.2. movement :

We call **periodic movement** a movement which repeats itself and in which each **cycle** reproduces identically.

1.3. Period :

The duration of a **cycle** is called **period** .

1.4. movement oscillatory :

This type of **periodic movement** is called **oscillation** or **oscillatory movement** (such as the oscillations of a mass connected to a spring, the movement of a pendulum).

2. Different types of oscillations:

The oscillation of a system can be free (undamped, damped) or forced (undamped, damped).

2.1. Free undamped oscillation:

It is a system in which oscillations exist without the intervention of external forces (free) and the vibrations do not attenuate (do not stop).

2.2. Damped free oscillation:

It is the same system as the first (i.e. free) except that this time the oscillations are attenuated.

2.3. Undamped forced oscillation:

The oscillations are caused by the intervention of one or more external forces and the oscillations are not damped.

2.4. Damped forced oscillation:

It is the same system that precedes but this time the oscillations are attenuated.

3. Generalized coordinates, constraints and degrees of freedom :

3.1. coordinates :

3.1.1. Definition :

The generalized coordinates are the set of real **independent variables** making it possible to describe and configure all the elements of the system at any time t.

3.1.2. Noticed :

- By choosing to use **generalized coordinates** (denoted q) in our calculations, we are sure to work with independent variables.
- The derivatives of the generalized coordinates with respect to time give what are called the **generalized velocities** (denoted $\dot{q}$).

3.1.3. General case :

3.1.3.1. Point object (0 dimension) :

The number of coordinates necessary in a three-dimensional space to locate a point object is equal to **3** : x, y, zin a Cartesian coordinate system.

3.1.3.2. Solid object (3 dimension) :

The number of coordinates necessary in a three-dimensional space to locate each point of a solid body is equal to **6** :

- **3** movement coordinates $(x_{cg}, y_{cg}, z_{cg})$to locate its center of gravity,
- **3** angles of rotation (Euler angles) around the three axes passing through its center of gravity to locate the position of each solid body point relative to the center of gravity (Φ, θ, Ψ).

3.2. Constraints (C):

Constraints are the mathematical relationships that link coordinates together. The number of constraints $(N\ C)$is the number of mathematical relations.

3.3. The number of degrees of freedom (NDL):

3.3.1. Definition :

The number of degrees of freedom is the **number of independent** generalized coordinates necessary to configure all elements of the system at any time.

3.3.2. General case :

To identify **the number** of degrees of freedom, the following rules are always valid :

3.3.2.1. For N particles:

The number of degrees of freedom for N particles is defined by:

$$Ndl = 3N - N\,C$$

3.3.2.2. For N solid body :

The number of degrees of freedom for N a solid body is defined by:

$$Ndl = 6N - N\,C$$

With :

Ndl: The number of degrees of freedom.

$N\,C$: The number of constraints.

N : The number of particles (solid body).

3.4. Examples:

Give and **identify** the number of degrees of freedom (independent generalized coordinates) in the following systems:

A. Simple pendulum:

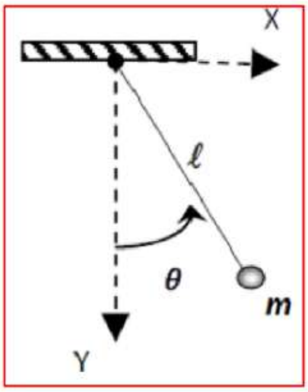
So $N\,C = 2$ we have : $Z = 0$ et $x^2 + y^2 = l$

And m: is a point mass, i.e. $N = 1$

We obtain :

$Ndl = 3N - N\,C = 3 * 1 - 2 = 1$

independent generalized coordinate is: θ.



B. Free particle:

We have : $N\,C = 0$ And the free particle is a point object. i.e. $N = 1$

We obtain :

$Ndl = 3N - N\,C = 3 * 1 - 0 = 3$

independent generalized coordinates are: x, y, z in a Cartesian frame of reference.

Noticed :

- ✓ To study an oscillatory system, you must take the following steps:
  1. Determination of the equation of motion.
  2. Derive the solution.
  3. Draw the curves.

- ✓ To determine the equation of motion we can use three methods:
  1. The fundamental law of dynamics.
  2. Theorem of conservation of mechanical energy.
  3. The Lagrange formalism $(L = T - V)$
     - T: kinetic energy .
     - V:energy .

4. *Kinetic energy(T) from a material point:*

4.1. Kinetic energy $(T)$for a linear movement:

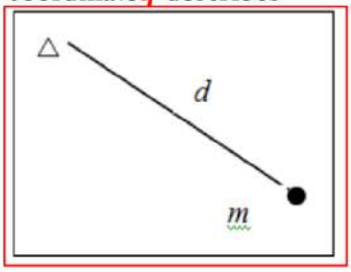
A material point can **move** in a straight line, so the coordinate q describes a movement. In this case the kinetic energy of a material point is written in the following form:

$$T = \frac{1}{2} m \dot{q}^2$$

m : The mass of the particle.
$\dot{q}$: speed **linear** .

4.2. Kinetic energy $(T)$for a rotation:

If, on the other hand, the material point rotates **around a fixed axis** , the coordinate$q$ describes **an angle of rotation** , and consequently kinetic energy is written



$$T = \frac{1}{2} m d^2 \dot{q}^2 \ldots (1)$$

$\dot{q}$:**angular** velocity .

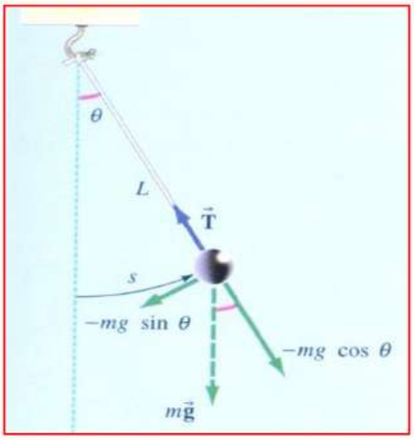
md^2: in a dimension of a **moment of inertia .**

It is **the moment of inertia** of a **point mass** which is at a distance $(d)$of the axis **of rotation** (Δ), we can write it in the form$(I_{(\Delta)})$.

We can write the equation (1)in the form:

$$T = \frac{1}{2} I_{(\Delta)}\, \dot{q}^{\,2}$$

Example : the kinetic energy of a simple pendulum:



$$T = \frac{1}{2} m\, \dot{q}^{\,2}$$

$$T = \frac{1}{2} m\, \dot{s}^{\,2}$$

Since :

$$T = \frac{1}{2} m\, \dot{s}^{\,2}$$

$$s = l\theta \Rightarrow \dot{s} = l\dot{\theta}$$

$$T = \frac{1}{2} m\, \dot{s}^{\,2} = \frac{1}{2} m\, (l\dot{\theta})^{\,2} = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$T = \frac{1}{2} I_{(\Delta)}\, \dot{\theta}^2$$

5. *Kinetic energy(T) for a solid body:*

Kinetic energy(T) for a solid body is defined by:

$$T = \frac{1}{2} I_{(\Delta)}\, \dot{q}^{\,2}$$

$I_{(\Delta)}$ **:** is the moment of inertia of the body relative to **the real axis of rotation** , we can identify $I_{(\Delta)}$ by two methods:

- **Direct method** : by identifying the moments of inertia of some solid body.
- **Indirect method** : By changing **the axis of rotation** (**Huygens ' theorem** or **parallel axes theorem**)

5.1. Direct method : Moment of inertia of some solid bodies (rotating):

Example : define the moment of inertia of a bar of homogeneous mass m, length l, and linear density λ, which rotates around an axis passing through:

a. the middle of the bar (cg).
b. its end (Δ).

We have: the linear density is defined by:

$$\lambda = \frac{M}{L} = \frac{dm}{dl} \Rightarrow dm = \lambda\, dl \ldots (1)$$

$$I = \int dm\, l^2 = \int \lambda\, dl\, l^2 \qquad \ldots (2)$$

a. The bar rotating around an axis passes through the middle of the bar (cg):

$$(2) \Rightarrow I_{(cg)=} \int_{-\frac{L}{2}}^{\frac{L}{2}} \lambda\, dl\, l^2 = \lambda \frac{L^3}{12} = \frac{ML^2}{12}$$

SO :

$$I_{(cg)=} \frac{ML^2}{12}$$

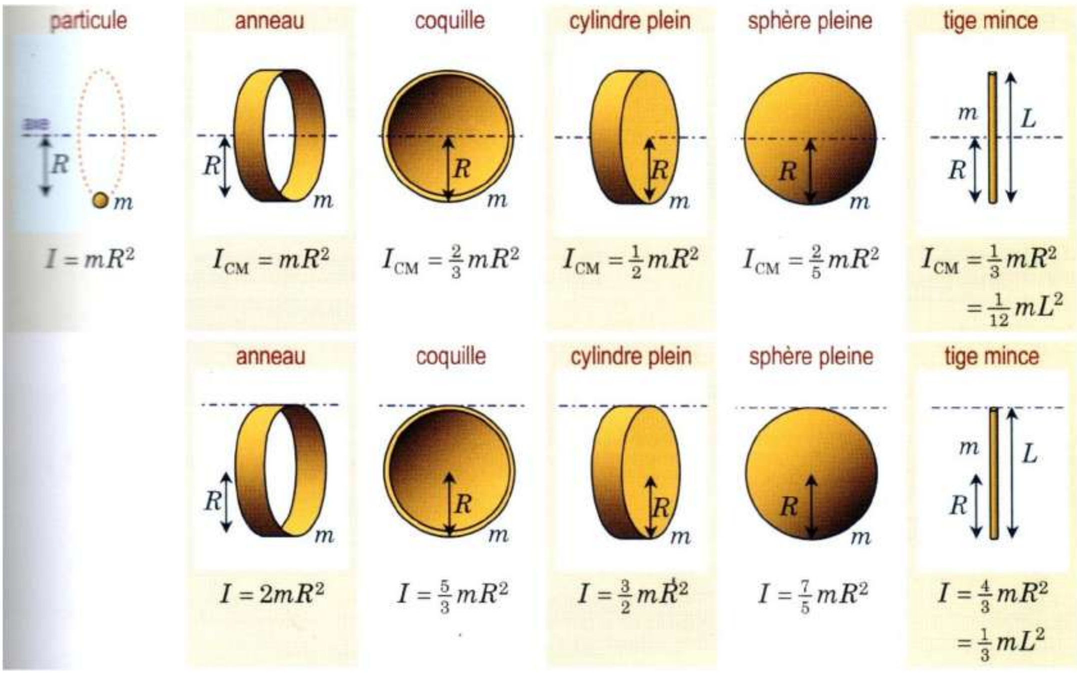
b. The bar rotating around an axis passes through the middle of the bar (Δ):

$$(2) \Rightarrow I_{(\Delta)=} \int_{0}^{L} \lambda\, dl\, l^2 = \lambda \frac{L^3}{3} = \frac{ML^2}{3}$$

SO :

$$I_{(\Delta)=} \frac{ML^2}{3}$$

By the same principle we can identify the moment of inertia of some solid bodies (the axis of rotation is identified by the broken line):

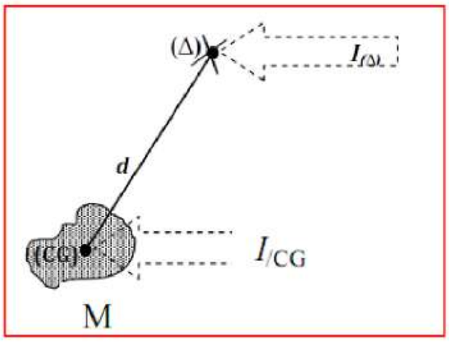


http://xn--webducation-dbb.com/moments-dinertie/

5.2. Indirect method : change of axis of rotation (Huygens' theorem or theorem of parallel axes):

We go from the moment of inertia around a straight line $(\Delta)$to the moment of inertia around a parallel axis but passing through the center of gravity $(cg)$of the solid (**figure below**), and vice versa, by the relation:

$$I_{(\Delta)}=I_{(cg)}+md^2$$



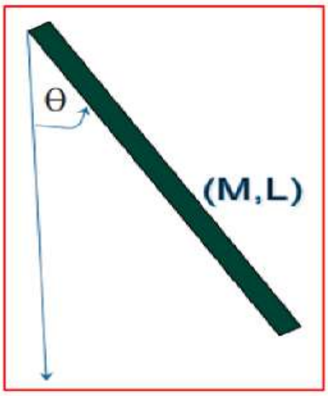
With :

- $I_{(\Delta)}$:Moment of inertia around a line (Δ).
- $I_{(cg)}$:Moment of inertia around the center of gravity(cg).
- d: The distance between the center of gravity and the actual axis of rotation (Δ).

Example :

Calculate **the kinetic energy** of a rod rotating around an axis passing through its end using direct calculation and indirect calculation.

- **Direct calculation:**



$$T=\frac{1}{2}I_{(\Delta)}\,\dot{q}^{\,2}\Rightarrow T=\frac{1}{2}I_{(\Delta)}\,\dot{\theta}^{\,2}$$

With :

$I_{(\Delta)}$: moment of inertia of a bar which rotates around an axis passing through its end (Δ), is defined by:

$$I_{(\Delta)}=\frac{ML^2}{3}$$

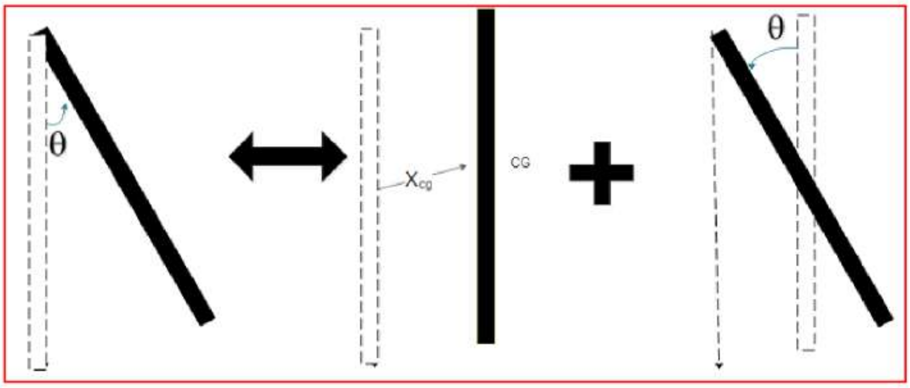
SO :

$$T=\frac{1}{6}ML^2\,\dot{\theta}^{\,2}$$

- **calculation :**

Using the Theorem of parallel axes

$I_{(\Delta)}=I_{(cg)}+md^2$



- $I_{(cg)}$:Moment of inertia around (cg)equal center of gravity:

  $I_{(cg)} = \frac{ML^2}{12}$
- d: The distance between the center of gravity and the actual axis of rotation(Δ) equal :

$$d = \frac{L}{2}$$

We obtain:

$$I_{(\Delta)}=I_{(cg)}+md^2 = \frac{ML^2}{12} + M(L/2)^2 = \frac{ML^2}{3}$$

this is the moment of inertia of a bar which rotates around an axis passing through its end (Δ), we obtain:

$$T = \frac{1}{6}ML^2\,\dot{\theta}^{\,2}$$

Noticed :

If replaces the formula for the moment of inertia deduced by the theorem of parallel axes $I_{(\Delta)}=I_{(cg)}+md^2$ in the kinetic energy formula$T = \frac{1}{2}I_{(\Delta)}\,\dot{\theta}^{\,2}$

We obtain :

$$T = \frac{1}{2}I_{(cg)}\,\dot{\theta}^{\,2} + \frac{1}{2}md^2\,\dot{\theta}^{\,2} \;\dots(1)$$

with : $d\dot{\theta} = v_{cg}$; v_{cg}: **there linear velocity** of center of gravity.

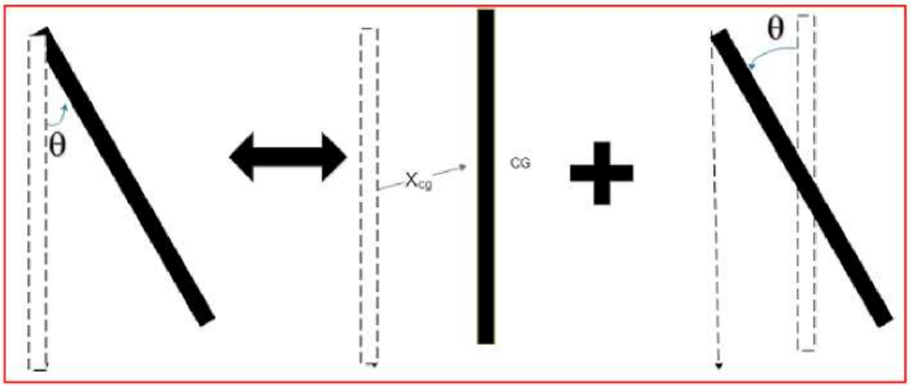
$$(1) \Rightarrow T = \frac{1}{2} I_{(cg)}\, \dot{\theta}^{\,2} + \frac{1}{2} m\, v_{cg}{}^{2} = T_{rot} + T_{tran}$$

- $T_{rot} = \frac{1}{2} I_{(cg)}\, \dot{\theta}^{\,2}$ **The pure rotational** kinetic energy of the bar around the center of gravity.
- $T_{tran} = \frac{1}{2} m\, v_{cg}{}^{2}$ **translational** kinetic energy of the center of gravity.

The total kinetic energy of a system of material points is equal to the sum of the translational kinetic energy of the center of gravity T_{tran} and the rotational kinetic energy about the center of gravity T_{rot}.

SO :

For the previous example and by using Huygens' theorem, we can write that the movement can be decomposed into **a rotation** around the center of gravity plus **a translation** of the center of severity **:** $T = T_{rot} + T_{tran} \dots (*)$



- $T_{rot} = \frac{1}{2} I_{(cg)}\, \dot{\theta}^{\,2}$ with : $I_{(cg)} = \frac{ML^2}{12}$; SO : $T_{rot} = \frac{ML^2}{24} \dot{\theta}^{\,2}$
- $T_{tran} = \frac{1}{2} M\, v_{cg}{}^{2}$ with: $v_{cg} = d\dot{\theta} = \frac{L}{2}\dot{\theta}$therefore : $T_{tran} = \frac{1}{2} M \frac{L^2}{4}\ \dot{\theta}^{\,2} = \frac{ML^2}{8} \dot{\theta}^{\,2}$

We obtain : $(*) \Rightarrow T = T_{rot} + T_{tran} = \frac{ML^2}{24} \dot{\theta}^{\,2} + \frac{ML^2}{8} \dot{\theta}^{\,2} = \frac{ML^2}{6} \dot{\theta}^{\,2}$

$$T = \frac{1}{6} ML^2\, \dot{\theta}^{\,2}$$

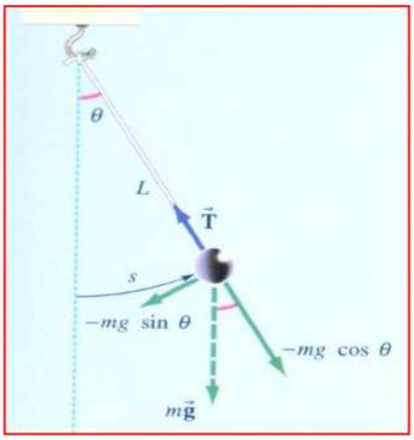
Special case: for a simple pendulum:

$I_{(\Delta)} = I_{(cg)} + md^2 \Rightarrow$

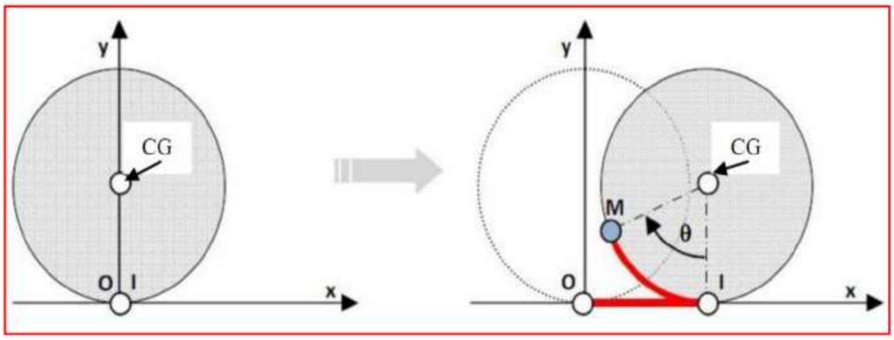
$$T_{(\Delta)} = \frac{1}{2} I_{(\Delta)} \dot{\theta}^{\,2} = \frac{1}{2} I_{(cg)}\, \dot{\theta}^{\,2} + \frac{1}{2} md^2\, \dot{\theta}^{\,2}$$
$$= \frac{1}{2} I_{(cg)}\, \dot{\theta}^{\,2} + \frac{1}{2} m\, v_{cg}{}^{2}$$
$$= T_{rot} + T_{tran}$$

Or : $I_{(cg)} = 0 \Rightarrow T_{rot} = 0$ for a point mass.

SO : $T_{(\Delta)} = T_{tran} = \frac{1}{2} ml^2 \dot{\theta}^2$



Example 02: give the formula for the kinetic energy of a disk which moves without sliding on a straight line.



$$I_{(\Delta)}=I_{(cg)}+md^2 \Rightarrow T_{(\Delta)} = \frac{1}{2} I_{(cg)}\,\dot{\theta}^{\,2} + \frac{1}{2} M\, v_{cg}^{\;2} = T_{rot} + T_{tran}$$

We have : $T_{rot} = \frac{1}{2} I_{(cg)}\,\dot{\theta}^{\,2}$

With $I_{(cg)}$ is the moment of inertia of the disk relative to the axis which passes through its **center of gravity.**

$$I_{(cg)} = \frac{1}{2} MR^2 \Rightarrow T_{rot} = \frac{1}{4} MR^2 \dot{\theta}^{\,2}$$

no- **slip rolling** condition gives us:

$$d = OI = IM = R\theta \Rightarrow \dot{d} = v_{cg} = R\dot{\theta}$$

Or: v_{cg} : the speed of the center of gravity.

SO :$T_{tran} = \frac{1}{2} M\, v_{cg}^{\;2} = \frac{1}{2} MR^2\dot{\theta}^2$

And consequently :

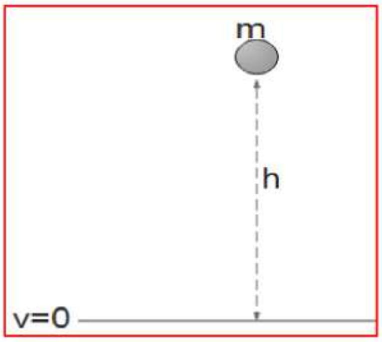
$$T_{(\Delta)} = T_{rot} + T_{tran} = \frac{1}{4} MR^2 \dot{\theta}^{\,2} + \frac{1}{2} MR^2\dot{\theta}^2 = \frac{3}{4} MR^2\dot{\theta}^2$$

6. *Potential energy* (V):

6.1. Gravitational potential energy:

This is the energy due to the **gravitational field** , and this energy depends on an **arbitrary choice** of the zero level of potential energy.

- Center of gravity is **above** the level $V = 0$.

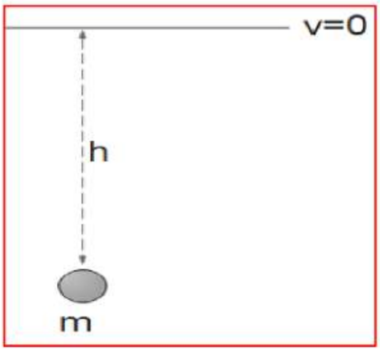


$$V = mgh + C$$

C: Is a constant that can be determined from an arbitrary choice of the zero level of potential energy.
h: Is the height of the mass relative to the zero level of potential energy.

For $h = 0$, we chose $V = 0$, and therefore $C = 0$.

- Center of gravity is **below** the level $V = 0$.



$$V = -mgh + C$$

In this case also for $h = 0$, we chose $V = 0$, and therefore $C = 0$

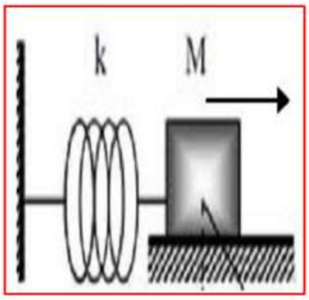
Noticed

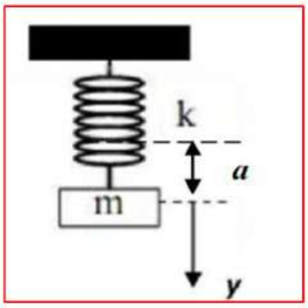
In calculations the observation Cdisappears by derivation or subtraction, because we are generally only interested in the difference between the potential energies associated with two points in space.

6.2. Elastic potential energy:

This is the potential energy stored in a spring for example, by extending or compressing it compared to its empty languor.

$$V = \frac{1}{2}K(allongement \text{ ou } compression)^2 + C$$



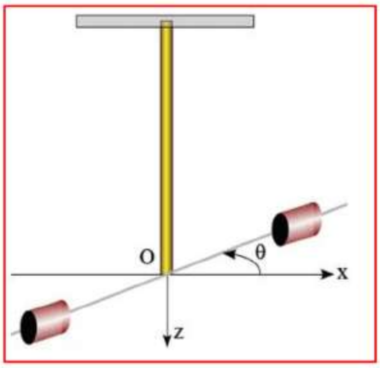


$V = \frac{1}{2}K(x)^2 + C$

$V = \frac{1}{2}K(y + a)^2 + C$

K: is the spring stiffness constant (N/m).
a: is the extension of the spring to the equilibrium position.

6.3. Torsion potential energy:



$$V = \frac{1}{2}C\theta^2$$

C: is the torsion constant (N/rad).

General case: The potential energy of a **linear oscillatory system** is proportional to the square of the variable q, i.e.:

$$V = \frac{1}{2}(cte)q^2$$

7. *Balance of a system:*

A system is said to be in a state of equilibrium if the resultant force exerted on the system is zero $\Sigma\vec{F} = \vec{0}$. The system is then either at rest, or it carries out a uniform rectilinear movement (this is the principle of inertia, see physics course 01 chapter 03).

For a conservative system $\vec{F} = -\overrightarrow{gradV}$, at the equilibrium position $(q_0)$the force is zero:

$$F = 0 \Rightarrow \left.\frac{\partial V}{\partial q}\right|_{q_0} = 0$$

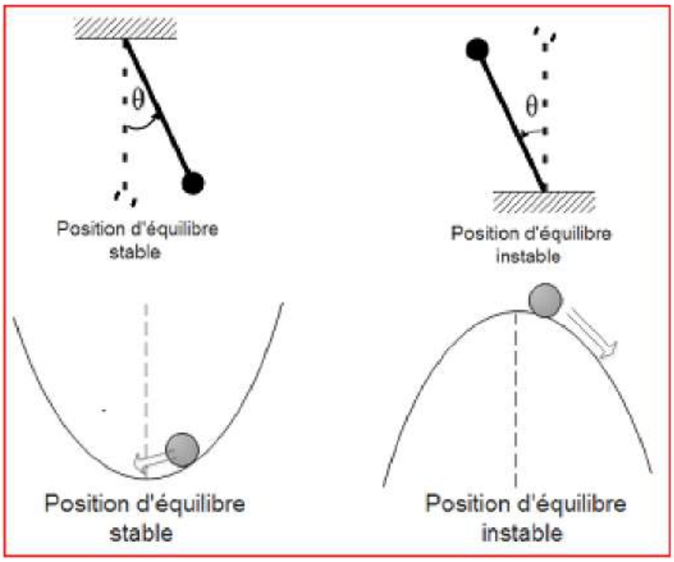
- This position corresponds to a **stable equilibrium position** when:

$$\left.\frac{\partial^2 V}{\partial q^2}\right|_{q_0} > 0$$

- **unstable equilibrium** position when:

$$\left.\frac{\partial^2 V}{\partial q^2}\right|_{q_0} < 0$$

Example :



For the simple pendulum in the figure above, the expression for **potential energy** is written: $V = -mg\cos(\theta)$, equilibrium $\left.\frac{\partial V}{\partial \theta}\right|_{\theta_0} = 0$, SO : $mg\sin(\theta) = 0 \Rightarrow \theta = 0$ ou $\theta = \pi$.

HAS :

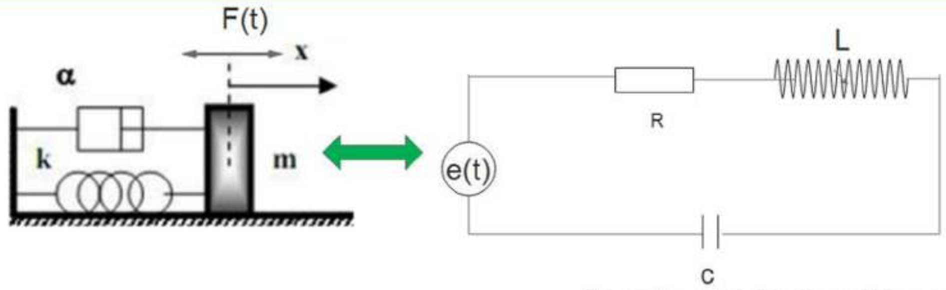
- $\theta = 0 \Rightarrow \left.\frac{\partial^2 V}{\partial \theta^2}\right|_{\theta=0} > 0$; SO HAS : $\theta = 0$; we have **stable equilibrium.**
- $\theta = \pi \Rightarrow \left.\frac{\partial^2 V}{\partial \theta^2}\right|_{\theta=\pi} < 0$; SO HAS : $\theta = \pi$; we have **unstable equilibrium.**

8. *Electromechanical analogy:*

Mechanical systems can be represented by analogous electrical circuits.

We see that the relationship between the electric current $\boldsymbol{i(t)}$and the electric charges $\boldsymbol{q}$is exactly of the same nature as that which exists between the speed $\boldsymbol{v(t)}$and the position $\boldsymbol{x}$of a mechanical system:

$$\boldsymbol{i(t) = \frac{dq}{dt}} \text{ And} \boldsymbol{v(t) = \frac{dx}{dt}}$$



En appliquant le formalisme de Lagrange:

$m\ddot{x} + \alpha\dot{x} + kx = F(t)$

En appliquant la loi des mailles, on trouve:

$L\ddot{q} + R\dot{q} + \frac{1}{C}q = e(t)$

Mechanical system		Electrical system
Shift : $x(t)$	**Corner** :$\theta(t)$	**Charge** :$q(t)$
Speed : $v(t) = \dot{x}$	**Angular velocity** :$\dot{\theta}(t)$	**Fluent** :$i(t) = \dot{q}$
Acceleration:$a(t) = \ddot{x}$	Angular acceleration$\ddot{\theta}(t)$	Current variation:$\dot{i}' = \ddot{q}$
Stiffness constant : k	**Torsion constant** :C	**Reverse of capacity** :$1/C$
Mass : m	Moment of inertia :I	**Coil** :L
Kinetic energy :$\frac{1}{2}m\dot{x}^2$	Kinetic energy :$\frac{1}{2}I\dot{\theta}^2$	Magnetic energy:$\frac{1}{2}L\dot{q}^2$
External strength : F_{ext}		**Tension** : $e(t)$
Shock absorber : α		**Resistance** :R
Return force: $k\,x$		**ddp between the terminals of a capacitor**q/C
Inertia force:$m\,\ddot{x}$		**ddp between the coil terminals:**$L\ddot{q}$
Potential energy :$\frac{1}{2}kx^2$		**Electric energy** : $\frac{1}{2C}q^2$

Chapter 2

Free oscillations with one degree of freedom.

1. Introduction :

For a free system , oscillations exist without the intervention of external forces and consequently friction forces are neglected. And the vibrations don't subside (don't stop).

- The identification of the position, speed and acceleration of each point of this body in relation to a chosen reference point is made by solving the equation of motion.

2. Lagrange formalism:

Around 1756, Joseph-Louis Lagrange introduced a function of dynamic variables which made it possible to write concisely the equations of motion of the system, then he developed analytical mechanics.

Joseph-Louis Lagrange
1707-1813
Mathématicien et Astronome Italien

This formalism is based on the Lagrange function $L = T - V$; the equation of motion is written in the form for a conservative system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = 0 \dots (1)$$

- L: Lagrange or Lagrangian function.
- T: The kinetic energy of the system.
- P: The potential energy of the system.
- q: The generalized coordinate.
- $\dot{q}$: Generalized speed.

- In the case of a speed-dependent friction force $F_q = -\alpha\dot{q}$.

- the equation (1) is written in the form:
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)=-\alpha\dot{q}$ Or $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)+\frac{\partial D}{\partial \dot{q}}=0 \dots (2)$; With $D$is the dissipation function given by $D=\frac{1}{2}\alpha\dot{q}^2$; it is linked to the friction force by: $F_q=-\frac{\partial D}{\partial \dot{q}}$

- In the case of an external time-dependent force in addition to the friction force, the equation (2) is written in the form: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)+\frac{\partial D}{\partial \dot{q}}=F(t)$; With: $F(t)$: external force.

3. Approximation of weak oscillations (amplitudes):

For weak oscillations, we apply the Taylor expansion:

If $x$is very small (i.e. tends to zero) the Taylor expansion around $x_0=0$, written in the following form:

- $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''^{(0)}+\cdots$

So for θvery small (i.e. tends to zero), we can write the equations $sin\ (\theta)$and $cos\ (\theta)$following Taylor 's development:

- $\sin(\theta)=\sin(0)+\theta\cos(0)+\frac{\theta^2}{2!}(-\sin(0))+\frac{\theta^3}{3!}\cos(0)+\cdots$
- $\cos(\theta)=\cos(0)+\theta(-\sin(0))+\frac{\theta^2}{2!}(-\cos(0))+\frac{\theta^3}{3!}\sin(0)+\cdots$

Generally we limit ourselves to the second term for the sine function and to the third term for the cosine function.

- $sin(\theta)\approx\theta$
- $cos(\theta)\approx 1-\frac{\theta^2}{2!}$ with θ in (rad).
  Example :

- $sin(5^0)=sin(.rad)=.$

4. Equations of motion of a free system :

To identify the equation of motion, three methods are generally used:

(a) The application of Newton's 2nd law (see physics course 01)

(b) The application of the principle of conservation of total energy or mechanical energy, This method is only applicable for isolated systems, where all forces are conservative (see physics course 01).

(c) The application of an extremely effective mathematical formalism called the Lagrange formalism, and this formalism which interests us in this chapter)

- **Example (01): the equation of motion of an elastic pendulum:**

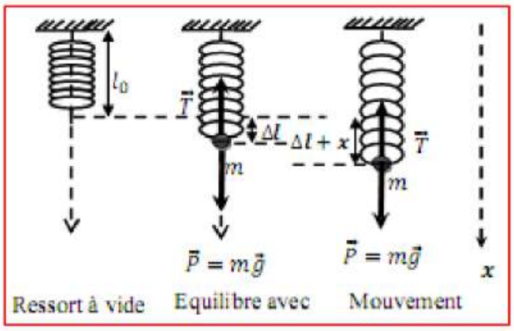
A- The equation of motion of a vertical elastic pendulum:

a) By application of the fundamental law of dynamics (LFD) :

- **At static equilibrium:**

$\Sigma \vec{F} = \vec{0}$

$\vec{P} + \vec{F} = \vec{0}$ With: $\vec{P}$: the weight of the mass m.
$\vec{F}$: the return force of the spring.



$mg - K\,x_0 = 0$ avec : $x_0 = l - l_0$

$mg = K\,x_0$ (This is the equilibrium condition)

- **At the move :**

$\Sigma \vec{F} = m\,\vec{a}$

$\vec{P} + \vec{F} = m\,\vec{a}$

$mg - K\,(x + x_0) = m\frac{d^2x}{d\,t^2} = m\,\ddot{x}$(According to static equilibrium $mg = K\,x_0$)

$-K\,x = m\,\ddot{x}$

$$\ddot{x} + \frac{K}{m}\,x = 0 \ldots (1)$$

b) By the method of mechanical energy conservation:

$$E_M = E_c + E_P = T + V = cte$$

SO : $\frac{d}{dt}(T + V) = 0$

$T = \frac{1}{2}m\dot{x}^2$

$V = V_K + V_g$

- $V_k = \frac{1}{2}k(x + x_0)^2$
- $V_g = -mg(x + x_0)$(We take as reference to:$x = l_0 \Rightarrow V = 0$

SO :

$$V=\frac{1}{2}k(x+x_0)^2-mg(x+x_0)$$

At static equilibrium:

$$\left.\frac{\partial v}{\partial x}\right|_{x=0}=0 \Rightarrow \frac{\partial}{\partial x}\left[V_K+V_g\right]=0$$

We find :$mg=K\,x_0$ **(** This is the equilibrium condition)

SO :

- $\frac{\partial E_T}{\partial t}=0$
- $\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2+\frac{1}{2}k(x+x_0)^2-mg(x+x_0)\right)=0$

(And according to static equilibrium $mg=K\,x_0$)

We obtain : $\ddot{x}+\frac{K}{m}x=0 \ldots(1)$

c) **By the Lagrange formalism:**

$L=T-V$

- $T=\frac{1}{2}m\dot{x}^2$
- $V=V_K+V_g=\frac{1}{2}k(x+x_0)^2-mg(x+x_0)$

SO :

$$L=\frac{1}{2}m\dot{x}^2-\frac{1}{2}k(x+x_0)^2+mg(x+x_0)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\left(\frac{\partial L}{\partial x}\right)=0$$

At static equilibrium:

$$\left.\frac{\partial v}{\partial x}\right|_{x=0}=0 \Rightarrow \frac{\partial}{\partial x}\left[V_K+V_g\right]=0$$

$$mg=K\,x_0$$

We obtain :

$$\ddot{x}+\frac{K}{m}x=0 \quad \ldots(1)$$

B- The equation of motion of a horizontal elastic pendulum:

a) By application of the fundamental law of dynamics (LFD) :

$V_g = 0$ And $x_0 = 0$

- Next: (**OY)** :

$\Sigma \vec{F} = \vec{0}$

$\vec{P} + \vec{R} = \vec{0}$

- Next: (**OX):**

$\Sigma \vec{F} = m\,\vec{a}$

$\vec{F} = m\,\vec{a}$

$-K\,x = m\,\ddot{x}\ \ (x_0 = 0) \Rightarrow \ddot{x} + \frac{K}{m}\,x = 0 \ldots (1)$

b) By the method of mechanical energy conservation:

$$T + V = cte \ \Rightarrow \frac{d}{dt}(T + V) = 0$$

- $T = \frac{1}{2} m\dot{x}^2$
- $V = V_K = \frac{1}{2} kx^2$

$$\frac{d}{dt}\left(\frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2\right) = 0$$

We obtain : $\ddot{x} + \frac{K}{m}\,x = 0 \ldots (1)$

c) By the Lagrange formalism:

$L = T - V$

- $T = \frac{1}{2} m\dot{x}^2$
- $V = V_K = \frac{1}{2} kx^2$

SO :

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0$$

We obtain : $\ddot{x} + \frac{K}{m}\, x = 0 \dots (1)$

5. Solution of the equation of motion:

Equation (1) is a 2nd order differential equation of the form

$\ddot{x} + w_0^2\, x = 0$, with $w_0^2 = \frac{K}{m}, T = 2\pi\sqrt{\frac{m}{k}}$.

The solution to this equation is of the form:

$$\begin{cases} x(t) = A\, e^{rt} \dots (2) \\ \dot{x}(t) = Are^{rt} \\ \ddot{x}(t) = Ar^2\, e^{rt} \dots (3) \end{cases}$$

, Or $A$and $r$are constants.

- We determine$r$ =?

The equation (2) and (3)in (1), we obtain:

$r^2 + w_0^2 = 0$ SO :$r_1 = iw_0$ And $r_2 = -iw_0$.

The general solution of $x(t)$is then a linear combination of the two solutions: $x_1(t)$and $x_2(t)$.

With: $x_1(t) = A_1\, e^{r_1 t}$and$x_2(t) = A_2\, e^{r_2 t}$

So the general solution is:

$$x(t) = x_1(t) + x_2(t) = A_1\, e^{iw_0 t} + A_2\, e^{-iw_0 t} \dots (4)$$

Where $A_1$and $A_2$are constants which are determined from the initial conditions.

We pose:

$A_1 = \frac{1}{2} C\, e^{i\varphi}AndA_2 = \frac{1}{2} C\, e^{-i\varphi}$

Equation **(4)** becomes:

$x(t) = \frac{1}{2} C\left[e^{i(w_0 t+\varphi)} + e^{-i(w_0 t+\varphi)}\right]$, since $(e^{i\varphi} = \cos\varphi + i \sin\varphi)$we find:

$$x(t) = C\cos(w_0 t + \varphi)$$

6. Determination of constants (C) And (φ) :

We determine (C) *and* (φ) according to x_0, $\dot{x}_0$; and w_0;from the initial conditions:

We have:

$$\begin{cases} x(t) = C\cos(w_0 t + \varphi) \\ \dot{x}(t) = -C\, w_0 \sin(w_0 t + \varphi) \end{cases}$$

HAS : $t = 0 \Rightarrow \begin{cases} x(0) = C\,cos(\varphi) \quad\quad = x_0 \;...(1) \\ \dot{x}(0) = -Cw_0\,sin(\varphi) = \dot{x}_0 \;...(2) \end{cases}$

$$\begin{cases} (1)^2 \Rightarrow C^2\cos^2(\varphi) = {x_0}^2 \\ (2)^2 \Rightarrow C^2{w_0}^2\sin^2(\varphi) = {\dot{x}_0}^2 \end{cases} \Rightarrow \begin{cases} C^2\cos^2(\varphi) = {x_0}^2 \quad ...(3) \\ C^2\sin^2(\varphi) = \dfrac{{\dot{x}_0}^2}{{w_0}^2} \quad ...(4) \end{cases}$$

Equation (3) plus equation (4), we obtain:

$$C^2\,[\cos^2(\varphi) + \sin^2(\varphi)] = {x_0}^2 + \frac{{\dot{x}_0}^2}{{w_0}^2}$$

With :

$$\cos^2(\varphi) + \sin^2(\varphi) = 1$$

We obtain :

$$C = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{w_0}\right)^2}$$

And from:

$$\frac{(2)}{(1)} \Rightarrow \frac{-Cw_0\,sin(\varphi)}{C\,cos(\varphi)} = \frac{\dot{x}_0}{x_0} \Rightarrow -tg(\varphi) = \frac{\dot{x}_0}{w_0 x_0}$$

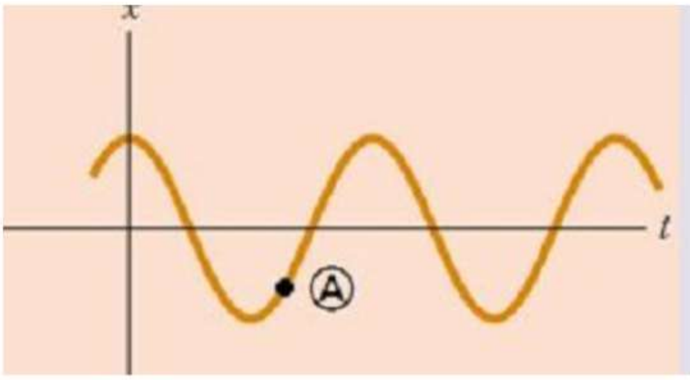
We find :

$$\varphi = tg^{-1}\left(\frac{-\dot{x}_0}{w_0\,x_0}\right)$$

(C) : is called the amplitude.
(φ) : The initial phase.

We can trace the curve of $x(t)$.

$$x(t) = C\,cos(w_0 t + \varphi)$$



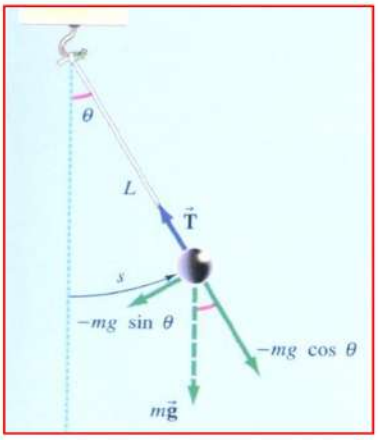
Periodic motion (harmonic oscillation)

Example (0 2): the equation of motion of a simple pendulum:

a) By application of the fundamental law of dynamics (LFD) :

$$\Sigma \vec{F} = m\,\vec{a}$$

- Next oy: $\vec{P} + \vec{T} = \vec{0}$ (we have no following movement oy)
- Next ox: $\vec{P} + \vec{T} = m\,\vec{a}$



$\vec{P}$: the weight of the mass m
$\vec{T}$: the thread tension.

$$-mg\sin\theta = m\frac{d^2S}{dt^2}$$

$$\sin\theta \approx \theta,\ s = L\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

$$\omega = \sqrt{\frac{g}{L}},\ T = 2\pi\sqrt{\frac{L}{g}}$$

b) By the Lagrange formalism or by conservation of mechanical energy:

$$E_c = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{s}^2 = \frac{1}{2}m(\ell\dot{\theta})^2 = \frac{1}{2}m\ell^2\dot{\theta}^2 = \frac{1}{2}I\dot{\theta}^2$$

$$car : s = \ell\theta$$

$$et : I = ml^2$$

$$Ep = mg\ell(1-\cos\theta)$$

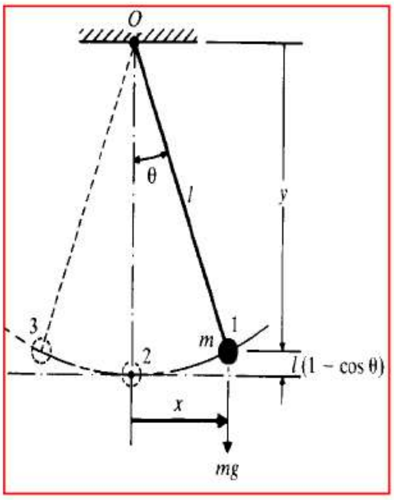
$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(\ell - \cos\theta)$$

$$m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0 \quad \Rightarrow$$

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

$$car \ \sin\theta \approx \theta, (faible\ oscilation)$$

$$\omega = \sqrt{\frac{g}{l}}, \ T = 2\pi\sqrt{\frac{l}{g}}$$



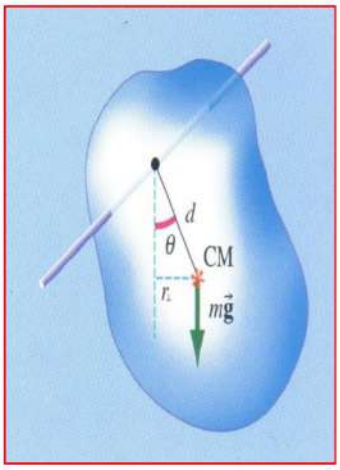
Example (03): the equation of motion of a compound pendulum:

$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

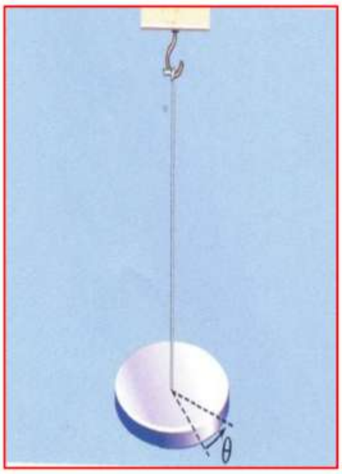
$$\sin\theta \approx \theta$$

$$\frac{d^2\theta}{dt^2} + \frac{mgd}{dt^2}\theta = 0$$

$$\omega = \sqrt{\frac{mgd}{I}}, \ T = 2\pi\sqrt{\frac{I}{mgd}}$$



Example (04): the equation of motion of a torsion pendulum:



$$T=\frac{1}{2}I\dot{\theta}^2 \quad , \quad V=\frac{1}{2}K_t\theta^2$$

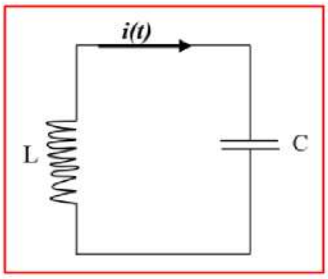
$$L=\frac{1}{2}I\dot{\theta}^2-\frac{1}{2}K_t\theta^2$$

$$I\ddot{\theta}+K\theta=0$$

$$\omega=\sqrt{\frac{K_t}{I}},\ T=2\pi\sqrt{\frac{I}{K_t}}$$

Example (05): Electrical system:

a- By Kirchhoff's law:



$$\boldsymbol{V_c+V_L=0\Rightarrow\frac{q}{C}+L\frac{di(t)}{dt}=0\ \dots(1)}$$

As $\boldsymbol{i(t)=\frac{dq}{dt}=\dot{q}}$, the differential equation of motion is written in the form:

$\boldsymbol{\frac{q}{C}+L\ddot{q}=0\Rightarrow\ddot{q}+\frac{1}{LC}q=0}$, this equation is of the form: $\boldsymbol{\ddot{q}+w_0^2q=0}$, with $\boldsymbol{w_0=\sqrt{\frac{1}{LC}}}$

b- By the Lagrange method:

$$\boldsymbol{T=E_c=E_{mag}=V_L\,dq=\int L\frac{di}{dt}\,dq=\int L\frac{di}{dt}\,dq=\int L\frac{dq}{dt}di=\int L\,idi=\frac{1}{2}Li^2=\frac{1}{2}L\dot{q}^2}$$

$$\boldsymbol{V=E_p=E_{elec}=V_c\,dq=\int\frac{q}{c}\,dq=\frac{1}{2c}q^2}$$

Hence: Lagrangian: $\boldsymbol{L=T-V=\frac{1}{2}L\dot{q}^2-\frac{1}{2c}q^2}$

SO : $\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)=0\Rightarrow L\ddot{q}+\frac{q}{C}=0\Rightarrow\ddot{q}+\frac{1}{LC}q=0}$ **;** This equation is of the form: $\boldsymbol{\ddot{q}+w_0^2q=0}$, with $\boldsymbol{w_0=\sqrt{\frac{1}{LC}}}$

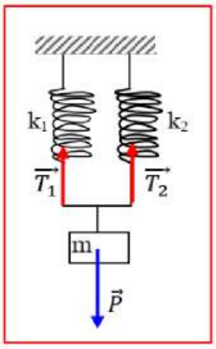
7. Equivalent stiffness constant:

Write the relationship between the equivalent stiffness constant and the associated stiffness constants for parallel and series mounting

- The two springs can be replaced by an equivalent spring with a stiffness constant: K_{eq};and an equivalent spring force.T_{eq}.

7.1.Parallel connection:

The figure schematizes the parallel association of two springs.



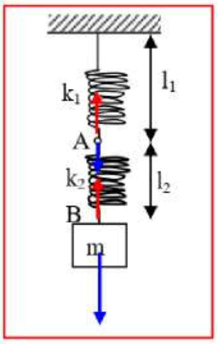
$$\Sigma \vec{F} = \vec{0} \Rightarrow \vec{P} = \vec{T_1} + \vec{T_2} = \overrightarrow{T_{eq}}$$

$$m\, g = K_1\, x + K_2\, x = K_{eq}\, x$$

We obtain :

$$K_{eq} = K_1 + K_2$$

7.2.Series installation:



The figure schematizes the series association of two springs.

Either :

x_1: The elongation of the spring K_1; such that : $m\, g = K_1\, x_1$

x_2: The elongation of the spring K_2; such that : $m\, g = K_2\, x_2$

$$x = x_1 + x_1 \Rightarrow m\, g \frac{1}{K_{eq}} = m\, g\left(\frac{1}{K_1} + \frac{1}{K_2}\right) \Rightarrow \frac{1}{K_{eq}} = (\frac{1}{K_1} + \frac{1}{K_2})$$

We obtain :

$$K_{eq} = \frac{K_1 K_2}{K_1 + K_2}$$

Problem corrected

Problem 1:

For the mechanical system shown to the right, the uniform rigid bar has mass m and pinned at point O. For this system:

a) find the equations of motion;

b) Identify the damping ratio and natural frequency in terms of the parameters m, c, k, and ℓ.

c) For:

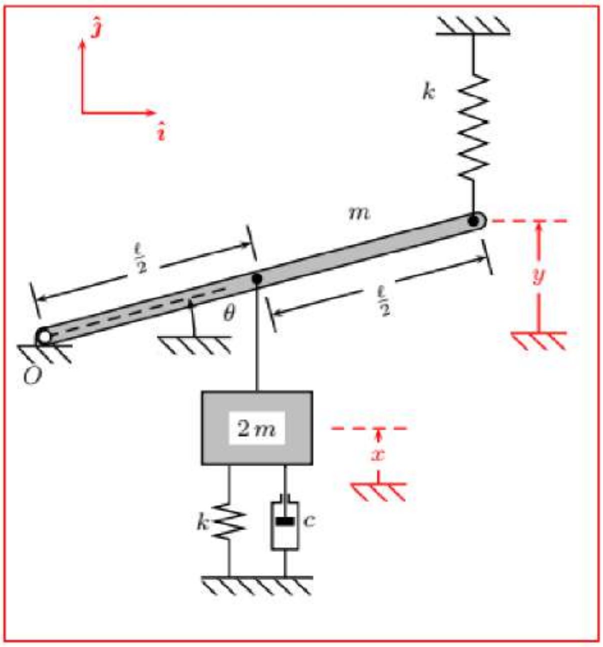
$$m = 1.50 \text{ kg}, \quad \ell = 45 \text{ cm},$$

$$c = 0.125 \text{ N/(m/s)}, \quad k = 250 \text{ N/m},$$

find the angular displacement of the bar $\theta(t)$ for the following initial conditions:

$$\theta(0) = 0,$$

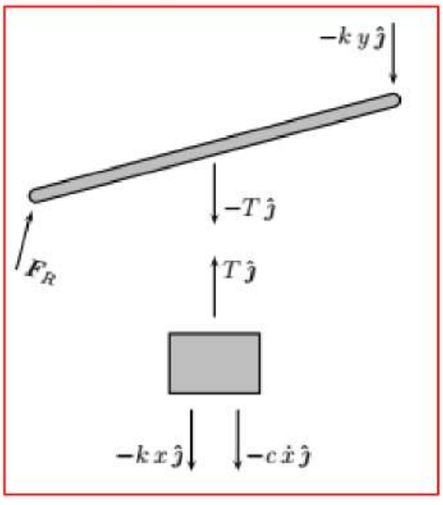
$$\dot{\theta}(0) = 10 \text{ rad/s}.$$



Assume that in the horizontal position the system is in static equilibrium and that all angles remain small.

Solution:

a) In addition to the coordinate θ identified in the original figure, we also define x and y as the displacment of the block and end of the bar respecively. The directions $\hat{\imath}$ and $\hat{\jmath}$ are defined as shown in the figure.
A free body diagram for this system is shown to the right. Note that the tension in the cable between the bar and the block is unknown and represented with T while the reaction force F_R is included, although both its magnitude and direction are



unspecified. In terms of the identifed coordinates, the angular acceleration of the bar $\boldsymbol{\alpha}_{\mathcal{B}/\mathcal{F}}$ and the

linear acceleration of the block $\boldsymbol{a}_G$ are

$$\boldsymbol{\alpha}_{\mathcal{B}/\mathcal{F}} = \ddot{\theta}\,\hat{\boldsymbol{k}}, \qquad {}^{\mathcal{F}}\mathbf{a}_G = \ddot{x}\,\hat{\boldsymbol{\jmath}}.$$

We can also relate the identified coordinates as

$$x = \frac{\ell}{2}\theta, \qquad y = \ell\,\theta.$$

The equations of motion for this system can be obtained with linear momentum balance applied to the block and angular momentum balance aout O on the bar. These can be written as

$$\begin{aligned} \sum \boldsymbol{F} = m\,{}^{\mathcal{F}}\mathbf{a}_G \longrightarrow \left(T - k\,x - c\,\dot{x}\right)\hat{\boldsymbol{\jmath}} &= 2\,m\,\ddot{x}\,\hat{\boldsymbol{\jmath}}, \\ \sum \boldsymbol{M}_O = I^O\,\boldsymbol{\alpha}_{\mathcal{B}/\mathcal{F}} \longrightarrow \left(-T\frac{\ell}{2} - k\,y\,\ell\right)\hat{\boldsymbol{k}} &= \frac{m\,\ell^2}{3}\,\ddot{\theta}\,\hat{\boldsymbol{k}}. \end{aligned}$$

Solving the first equation for T and substituting into the second equation yields

$$-\left(2\,m\,\ddot{x} + k\,x + c\,\dot{x}\right)\frac{\ell}{2} - k\,y\,\ell = \frac{m\,\ell^2}{3}\,\ddot{\theta}.$$

Using the coordinate relations we can obtain the equation of motion as

$$\frac{5\,m\,\ell^2}{6}\,\ddot{\theta} + \frac{c\,\ell^2}{4}\,\dot{\theta} + \frac{5\,k\,\ell^2}{4}\,\theta = 0.$$

b) In the above equation the equivalent mass, damping, and stiffness are

$$m_{\text{eq}} = \frac{5\,m\,\ell^2}{6}, \qquad b_{\text{eq}} = \frac{c\,\ell^2}{4}, \qquad k_{\text{eq}} = \frac{5\,k\,\ell^2}{4}.$$

From these the damping ratio and natural frequency are

$$\zeta = \frac{b_{\text{eq}}}{2\sqrt{k_{\text{eq}}\,m_{\text{eq}}}} = \frac{\frac{c\,\ell^2}{4}}{2\sqrt{\frac{5\,k\,\ell^2}{4}\,\frac{5\,m\,\ell^2}{6}}} = \frac{\sqrt{3}\,c}{2\sqrt{50\,k\,m}},$$

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m_{\text{eq}}}} = \sqrt{\frac{\frac{5\,k\,\ell^2}{4}}{\frac{5\,m\,\ell^2}{6}}} = \sqrt{\frac{3\,k}{2\,m}}$$

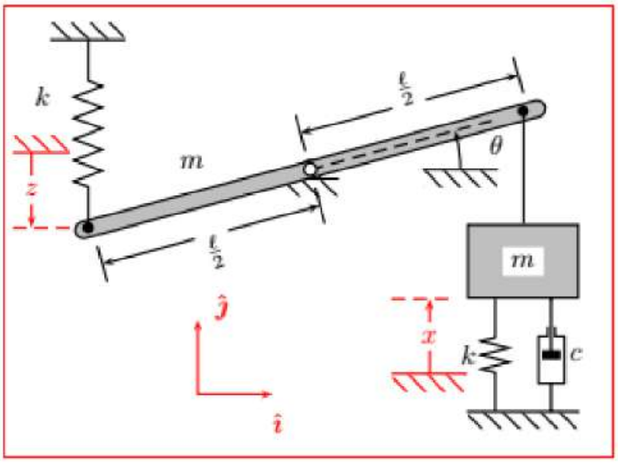
$$\theta(0) = b = 0, \qquad \dot{\theta}(0) = -\zeta\,\omega_n\,b + \omega_n\sqrt{1-\zeta^2}\,a = 10\ \text{rad/s},$$

so that the general solution becomes

$$\theta(t) = 0.632\,e^{-0.0125\,t}\sin\left(15.8\,t\right).$$

Problem 2:

For the mechanical system shown to the right, the uniform rigid bar has mass *m* and pinned at point *O*. For this system:



a) find the equations of motion;

b) Identify the damping ratio and natural frequency in terms of the parameters *m*, *c*, *k*, and ℓ.

c) For:

$$m = 2 \text{ kg}, \quad \ell = 25 \text{ cm},$$

$$c = 0.25 \text{ N/(m/s)}, \quad k = 50 \text{ N/m},$$

find the angular displacement of the bar $\theta(t)$ for the following initial conditions:

$$\theta(0) = 0, \dot{\theta}(0) = 10 \text{ rad/s}.$$

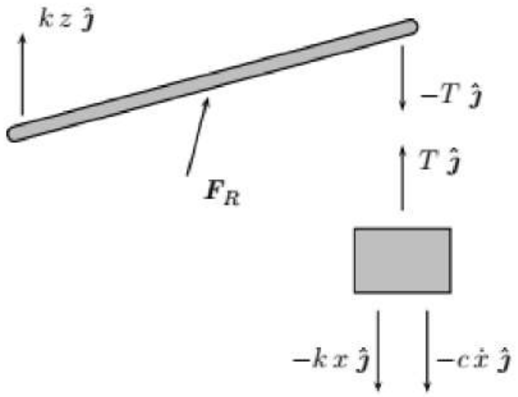
d) for this motion, find the tension in the cable connecting the rod and the block as a function of time. Assume that the system is in static equilibrium at $\theta = 0$, and that all angles remain small.

Solution:

a) We identify the coordinates x and z as shown above, which are related to the angular displacement **θ** as:

$$x = \frac{\ell}{2}\theta, \qquad z = \frac{\ell}{2}\theta.$$

An appropriate free-body diagram is shown to the right. Applying linear momentum balance on the block yields



$$\sum \boldsymbol{F} = m\,{}^{\mathcal{F}}\mathbf{a}_G,$$

$$(T - k\,x - c\,\dot{x})\,\hat{\jmath} = m\,\ddot{x}\,\hat{\jmath}.$$

Likewise, angular momentum balance on the bar provides

$$\sum \boldsymbol{M}_O = I^O\,\boldsymbol{\alpha}_{\mathcal{B}/\mathcal{F}},$$

$$\left(-T\frac{\ell}{2} - k\,z\,\frac{\ell}{2}\right)\hat{\boldsymbol{k}} = \frac{m\,\ell^2}{12}\,\ddot{\theta}\,\hat{\boldsymbol{k}}.$$

Combining these equations and eliminating the tension, the equation of motion can be written as

$$\frac{7\,m}{6}\,\ddot{\theta} + c\,\dot{\theta} + 2\,k\,\theta = 0.$$

b) For the above equation the equivalent mass, damping, and stiffness are

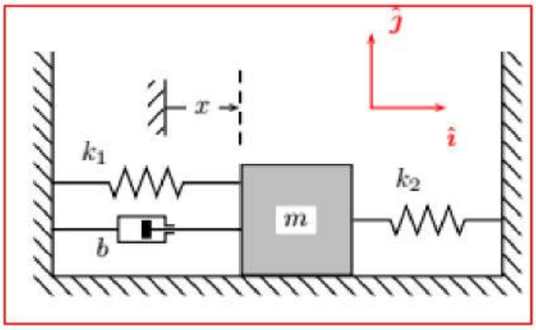
$$m_{\text{eq}} = \frac{7\,m}{6}, \qquad b_{\text{eq}} = b, \qquad k_{\text{eq}} = 2\,k,$$

and the natural frequency and damping ratio are

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m_{\text{eq}}}} = \sqrt{\frac{12\,k}{7\,m}}, \qquad \zeta = \frac{b_{\text{eq}}}{2\sqrt{k_{\text{eq}}\,m_{\text{eq}}}} = \frac{\sqrt{3}\,b}{\sqrt{28k\,m}}.$$

Problem 3:

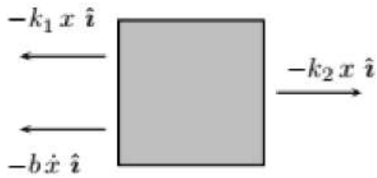
The block shown to the right rests on a frictionless surface. Find the response of the sys- tem if the block is displaced from its static equilibrium position 15 cm to the right and released from rest.



$m = 4.0$ kg, $\quad b = 0.25$ N/(m/s),
$k_1 = 1.5$ N/m, $\quad k_2 = 0.50$ N/(m/s).

Solution:

An appropriate free-body diagram is shown to the right. Notice that the two springs are effectively in parallel, as the displacement across each spring is identical. Linear momentum balance on this block provides



$$\begin{aligned} \sum \boldsymbol{F} &= m\,{}^{\mathcal{F}}\mathbf{a}_G, \\ (-k_1\,x - k_2\,x - b\,\dot{x})\,\hat{\imath} &= m\,\ddot{x}\,\hat{\imath}, \end{aligned}$$

or, writing this in standard form

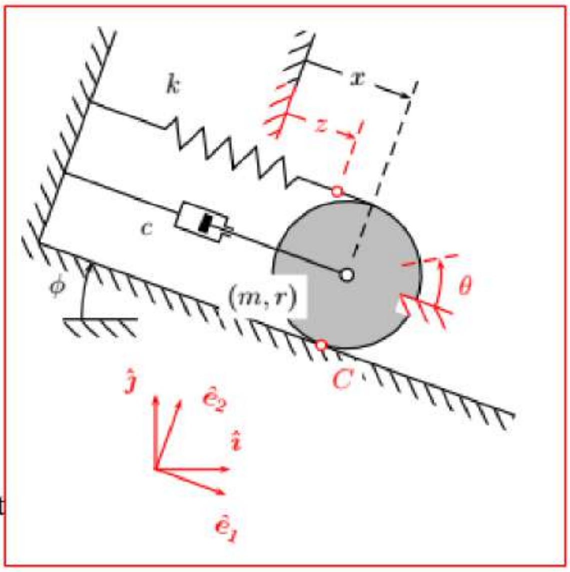
$$m\,\ddot{x} + b\,\dot{x} + (k_1 + k_2)\,x = 0.$$

Further, the system is released from rest so that the initial conditions are

$$x(0) = x_0 = 15\text{ cm}, \qquad \dot{x}(0) = 0\text{ cm/s}.$$

Problem 4:

For the system shown to the right, the disk of mass *m* rolls without slip and *x* measures the displacement of the disk from the unstretched position of the spring. The surface is inclined at an angle of φ with respect to vertical.



a) find the equations of motion. Do not neglect gravity;

b) if the system is underdamped, what is the frequency of the free vibrations of this system in terms of the parameters *k*, *c*, and *m*;

c) for what value of the damping constant *c* is the system critically damped;

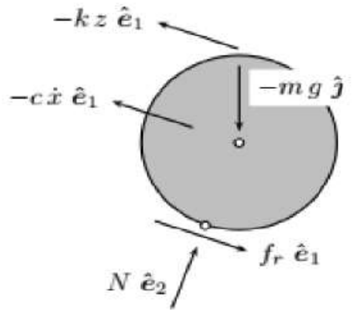
d) what is the static equilibrium displace- ment of the disk?

Solution:

a) In addition to x, the displacement of the center of the disk, we identify the coordinates z and θ, the displacement across the spring and the rotation of the disk respectively.

These additional coordinates are related to x as

$$z = 2\,x, \qquad x = -r\,\theta.$$



An appropriate free-body diagram is shown to the right. We note that $(\hat{\imath}, \hat{\jmath})$ are related to the directions $(\hat{e}_1, \hat{e}_2)$ as

$$\begin{aligned} \hat{\imath} &= \cos\phi\;\hat{e}_1 + \sin\phi\;\hat{e}_2, \\ \hat{\jmath} &= -\sin\phi\;\hat{e}_1 + \cos\phi\;\hat{e}_2. \end{aligned}$$

The moment produced by gravity about point C is

$$\begin{aligned} \boldsymbol{M}_{\text{gravity}} &= \mathbf{r}_{GC} \times (-m\,g\;\hat{\jmath}), \\ &= (r\;\hat{e}_2) \times (-m\,g\;\hat{\jmath}) = -m\,g\,r\,\sin\phi\;\hat{\boldsymbol{k}}. \end{aligned}$$

Angular momentum balance about the contact point C yields

$$\begin{aligned} \sum \boldsymbol{M}_C &= I^C\,\boldsymbol{\alpha}_{D/\mathcal{F}}, \\ \Big((2\,r)\,k\,z + r\,c\,\dot{x} - m\,g\,r\,\sin\phi\Big)\,\hat{\boldsymbol{k}} &= \left(\frac{3\,m\,r^2}{2}\,\ddot{\theta}\right)\hat{\boldsymbol{k}}. \end{aligned}$$

Eliminating the coordinates z and θ, we can write the equation of motion in terms of x as

$$\frac{3\,m}{2}\,\ddot{x} + c\,\dot{x} + 4\,k\,x = m\,g\,\sin\phi.$$

Since the gravitational force has been included in the development of this equation of motion, the coordinates are measured with respect to the unstretched position of the spring.

b) Assuming the system is underdamped, the frequency of the free vibrations is $\omega_d = \omega_n\,\sqrt{1-\zeta^2}$, where

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m_{\text{eq}}}} = \sqrt{\frac{8\,k}{3\,m}}, \qquad \zeta = \frac{b_{\text{eq}}}{2\sqrt{k_{\text{eq}}\,m_{\text{eq}}}} = \frac{c}{2\sqrt{6\,k\,m}},$$

so that

$$\omega_d = \sqrt{\frac{8\,k}{3\,m}}\sqrt{1-\frac{c^2}{24\,k\,m}},$$

c) The system is critically damped when $\zeta = 1$, which corresponds to a damping coefficient of

$$c_{\text{cr}} = 2\sqrt{6\,k\,m}.$$

d) The system is stationary in static equilibrium, so that $x \equiv x_0 =$ constant—both $\dot{x}$ and $\ddot{x}$ vanish, and the equation of motion reduces to

$$4\,k\,x_0 = m\,g\,\sin\phi.$$

Solving for x_0, the equilibrium displacement is

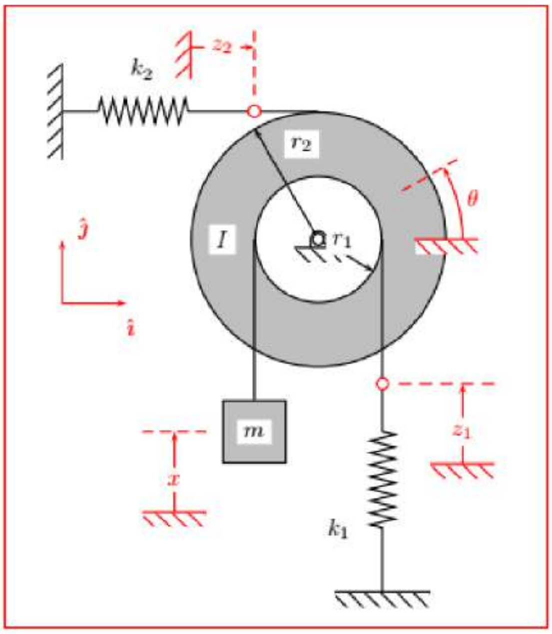
$$x_0 = \frac{m\,g\,\sin\phi}{4\,k}.$$

Problem 5:

In the figure shown to the right, in the absence of gravity the springs are unstretched in the equilibrium position.

a) Determine the deflection of each spring from its unstretched length when the system shown is in equilibrium.

b) If the system is released from the unstretched position of the springs, what is the maximum angular velocity of the disk during the resulting motion?

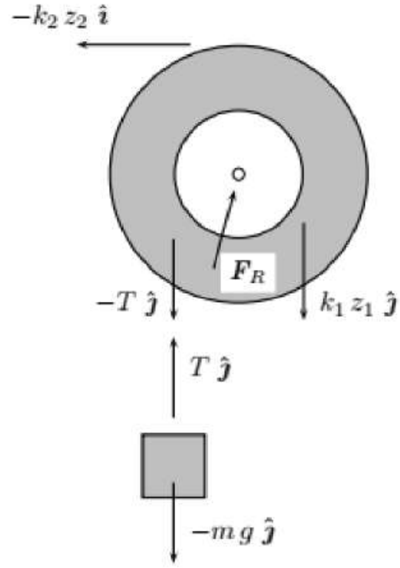


Solution:

a) We define the coordinates x, θ, z_1, and z_2 as shown in the figure, which are related as

$$x = -r_1\,\theta, \quad z_1 = r_1\,\theta, \quad z_2 = -r_2\,\theta.$$

Notice that because of these coordinate definitions, a rotation with positive θ gives rise to a negative value in both x and z_2. Likewise, we see that $x = -z_1$. Using the free-body diagram shown to the right, linear momentum balance on the block provides



$$\begin{aligned}\sum \boldsymbol{F} &= m\,{}^{\mathcal{F}}\mathbf{a}_G,\\ (T - m\,g)\,\hat{\jmath} &= m\,\ddot{x}\,\hat{\jmath},\end{aligned}$$

while angular momentum balance on the disk yields

$$\begin{aligned}\sum \boldsymbol{M}_O &= I^O\,\boldsymbol{\alpha}_{D/\mathcal{F}},\\ (T\,r_1 - k_1\,z_1\,r_1 + k_2\,z_2\,r_2)\,\hat{\boldsymbol{k}} &= I\,\ddot{\theta}\,\hat{\boldsymbol{k}}.\end{aligned}$$

Eliminating the unknown tension T from these equations and using the coordinate relations, the equation of motion becomes

$$\left(I + m\,r_1^2\right)\ddot{\theta} + \left(k_1\,r_1^2 + k_2\,r_2^2\right)\theta = m\,g\,r_1.$$

The equilibrium rotation of the disk thus is found to be

$$\theta_{\text{eq}} = \frac{m\,g\,r_1}{k_1\,r_1^2 + k_2\,r_2^2}.$$

With this, the equilibrium deflection of each spring is found to be

$$\begin{aligned}z_{1,\text{eq}} = r_1\,\theta_{\text{eq}} &= \frac{m\,g\,r_1^2}{k_1\,r_1^2 + k_2\,r_2^2},\\ z_{2,\text{eq}} = -r_2\,\theta_{\text{eq}} &= -\frac{m\,g\,r_1\,r_2}{k_1\,r_1^2 + k_2\,r_2^2}.\end{aligned}$$

b) The general free response of the disk can be expressed as

$$\theta(t) = \theta_{\text{eq}} + A\,\sin(\omega_n\,t) + B\,\cos(\omega_n\,t),$$

where θ_{eq} is given above, A and B are arbitrary constants, and

$$\omega_n = \sqrt{\frac{k_1\, r_1^2 + k_2\, r_2^2}{I + m\, r_1^2}}.$$

The system is released with the initial conditions:

$$\theta(0) = 0, \qquad \dot{\theta}(0) = 0,$$

so that solving for the arbitrary constants

$$A = 0, \qquad B = -\theta_{\text{eq}}.$$

Therefore the solution is

$$\theta(t) = \theta_{\text{eq}}\,(1 - \cos(\omega_n\, t)) = \frac{m\, g\, r_1}{k_1\, r_1^2 + k_2\, r_2^2}\left(1 - \cos\left(\sqrt{\frac{k_1\, r_1^2 + k_2\, r_2^2}{I + m\, r_1^2}}\, t\right)\right).$$

The angular velocity of the disk becomes

$$\dot{\theta}(t) = \left(\theta_{\text{eq}}\, \omega_n\right) \sin(\omega_n\, t),$$

which has amplitude

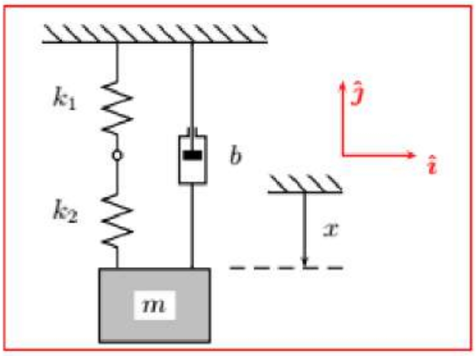
$$\Omega = \theta_{\text{eq}}\, \omega_n = \frac{m\, g\, r_1}{\sqrt{(k_1\, r_1^2 + k_2\, r_2^2)\,(I + m\, r_1^2)}}$$

Problem 6:

Find the response of the system shown to the right if the block is pulled down by 15 cm and released form rest.

$m = 2.0$ kg, $\qquad b = 0.5$ N/(m/s),

$k_1 = 0.5$ N/m, $\qquad k_2 = 0.25$ N/(m/s).

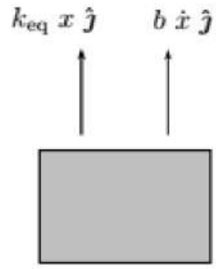


Solution:

For this system, the two springs in series may be replaced by an equivalent spring, with constant:

$$k_{\text{eq}} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} = \frac{k_1\, k_2}{k_1 + k_2}.$$

Therefore, the free-body diagram is shown to the right. Applying linear momentum balance to the block yields:



$$\begin{aligned} \sum \boldsymbol{F} &= m\, {}^{\mathcal{F}}\mathbf{a}_G, \\ \left(k_{\text{eq}}\, x + b\, \dot{x}\right)\, \hat{\jmath} &= -m\, \ddot{x}\, \hat{\jmath}, \end{aligned}$$

which can finally be written as:

$$m\, \ddot{x} + b\, \dot{x} + k_{\text{eq}}\, x = 0.$$

With the numerical values given above, this becomes:

$$(2\ \text{kg})\ \ddot{x} + \left(\frac{1}{2}\ \frac{\text{N}}{\text{m/s}}\right)\ \dot{x} + \left(\frac{1}{6}\ \frac{\text{N}}{\text{m}}\right)\ x = 0, \qquad x(0) = \frac{3}{20}\ \text{m}, \quad \dot{x}(0) = 0\ \text{m/s}.$$

With this, the damping ratio and natural frequency are:

$$\omega_n = \sqrt{\frac{1}{12}}\ \text{s}^{-1}, \qquad \zeta = \frac{\sqrt{3}}{4}.$$

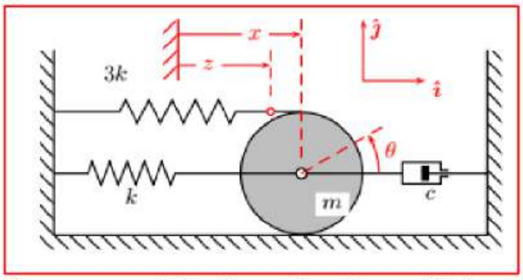
Therefore, the system is underdamped and the general response can be written as:

$$x(t) = e^{-\zeta\, \omega_n\, t}\left(A\ \cos(\omega_d\, t) + B\ \sin(\omega_d\, t)\right).$$

Using the initial conditions to solve for A and B, we find:

$$x(t) = \frac{3}{20}\ e^{-t/8}\left(\cos\left(\frac{\sqrt{13}}{8\sqrt{3}}\, t\right) + \sqrt{\frac{3}{13}}\cos\left(\frac{\sqrt{13}}{8\sqrt{3}}\, t\right)\right).$$

Problem 7:



For the system shown to the right, the disk of mass m rolls without slip and x measures the displacement of the disk from the unstretched position of the spring.

a) find the equations of motion;

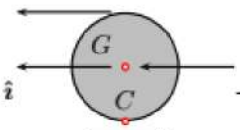
b) if the system is underdamped, what is the frequency of the free vibrations of this system in terms of the parameters k, c, and m;

Solution:

a) We define the three coordinates as shown as the figure, related as:

$$x = -r\ \theta. \qquad z = -2\ r\ \theta. \qquad z =$$

A free-body diagram for this system is shown to the right. Notice that the force in the upper spring depends on z, rather than x, while the friction force has an unknown magnitude f. Because the disk is assumed to roll without slip, we are unable to specify the value of f, but instead can relate the displacement and rotation of the disk through the coordinate relations above.



The equations of motions can be developed directly with angular momentum balance about the contact point, so that:

$$\sum \boldsymbol{M}_C = I^C\ \ddot{\theta}\ \hat{\boldsymbol{k}},$$

$$\Big((3k\ z)\ 2r + (k\ x)\ r + (c\ \dot{x})\ r\Big)\ \hat{\boldsymbol{k}} = \frac{3\ m\ r^2}{2}\ \ddot{\theta}\ \hat{\boldsymbol{k}}.$$

Finally, writing this equation in terms of a single coordinate, we obtain:

$$\left(\frac{3\ m\ r^2}{2}\right)\ \ddot{\theta} + (c\ r^2)\ \dot{\theta} + \left(13\ k\ r^2\right)\ \theta = 0.$$

b) For an underdamped response, the frequency of oscillation is $\omega_d = \omega_n\sqrt{1-\zeta^2}$. With this system, we find that:

$$\omega_n = \sqrt{\frac{26\ k}{3\ m}}, \qquad \zeta = \frac{c}{\sqrt{78\ k\ m}},$$

so that:

$$\omega_d = \sqrt{\frac{26\ k}{3\ m} - \frac{2\ c^2}{9\ m^2}}.$$

Chapter 3

Oscillations damped to one degree of freedom.

1. Introduction :

For a free system, harmonic oscillations never stop; But in reality the movement is carried out in a fluid (generally air) where there are always viscous friction forces, the oscillatory movement is then damped and ends up stopping.

2. Free oscillations damped to one degree of freedom:

For viscous friction, the friction force is proportional to the speed and implies a dissipation of energy in the form of heat, this energy dissipation is the main cause of motion damping.

The expression for the viscous friction force is given by:

$$F_q = -\alpha\dot{q}$$

Such as :

α : is the coefficient of viscous friction. its size is MT^{-1}
q : the generalized coordinate of the system.
$\dot{q}$: the generalized speed of the system.
The minus sign (-) comes from the fact that this force opposes the movement.

- Under the action of friction forces, the system dissipates (loses) mechanical energy in the form of heat, there is therefore a relationship between the force F_q and the dissipation function D on one side and the dissipation function and the coefficient of viscous frictionα .

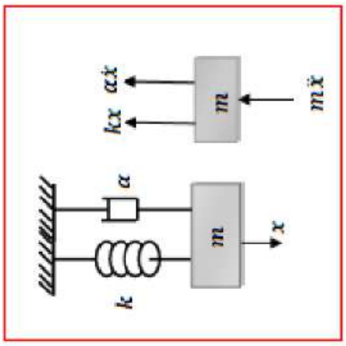
$$F_q = -\frac{\partial D}{\partial \dot{x}} \text{ And } D = \frac{1}{2}\alpha\dot{x}^2$$

3. Equation of motion of a damped system:

The Lagrange equation of a damped system with one degree of freedom is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = -\frac{\partial D}{\partial \dot{q}}$$

Equation of the movement of an elastic pendulum:



- The kinetic energy of the system: $T = \frac{1}{2}m\dot{x}^2$
- The potential energy of the system: $V = \frac{1}{2}kx^2$
- The dissipation function: $D = \frac{1}{2}\alpha\dot{x}^2$

- The Lagrange function:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = -\frac{\partial D}{\partial \dot{x}} \;...\;(1)$$

Or $:L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\,\ddot{x} \\ \left(\frac{\partial L}{\partial x}\right) = -K\,x \;...\;(2) \\ \frac{\partial D}{\partial \dot{x}} = \alpha\dot{x} \end{cases}$$

Substituting equation **(2)** in **(1)** We obtain :

$$m\ddot{x} + \alpha\dot{x} + k\,x = 0$$

$\ddot{x} + \frac{\alpha}{m}\,\dot{x} + \frac{K}{m}\,x = 0$ The form of the differential equation is $\ddot{x} + 2\lambda\dot{x} + w_0^2\,x = 0$

Or :

- λ: is called **the damping factor** , with: $\lambda = \frac{\alpha}{2m}$.
- w_0: is **the system's own pulsation** , with: $w_0^2 = \frac{K}{m}$.
- Either $\xi = \frac{\lambda}{w_0}$ (unitless), is **the damping ratio.**

We note that λ is homogeneous in the inverse of a time and its unit is s^{-1}. More λ is larger, the greater the damping and the faster the oscillator stops.

Solution of the equation of motion:

The differential equation of motion $\ddot{x} + 2\lambda\dot{x} + w_0^2\,x = 0 \;...\;(1)$ is a second-order linear differential equation with constant coefficients without a right-hand side.

The solution to this equation is of the form:

$$\begin{cases} x(t) = C\, e^{rt} & \dots (2) \\ \dot{x}(t) = Cr\, e^{rt} \\ \ddot{x}(t) = Cr^2\, e^{rt} \dots (3) \end{cases}$$

Equation (2) and (3) in (1), we obtain: $r^2 + 2\lambda\, r + w_0^2 = 0 \dots (4)$. The solution depends on the discriminant $\Delta = \lambda^2 - w_0^2$.

We distinguish three possible cases, depending on whether λ; is greater than, less than or equal to w_0:

4.1. First case: over-damped or aperiodic system (strong damping):

$$\Delta > 0 \Rightarrow (\lambda^2 - w_0^2) > 0 \Rightarrow \lambda > w_0 \text{ ou } \xi > 1$$

In this case, the characteristic equation (4) has **two real roots** :

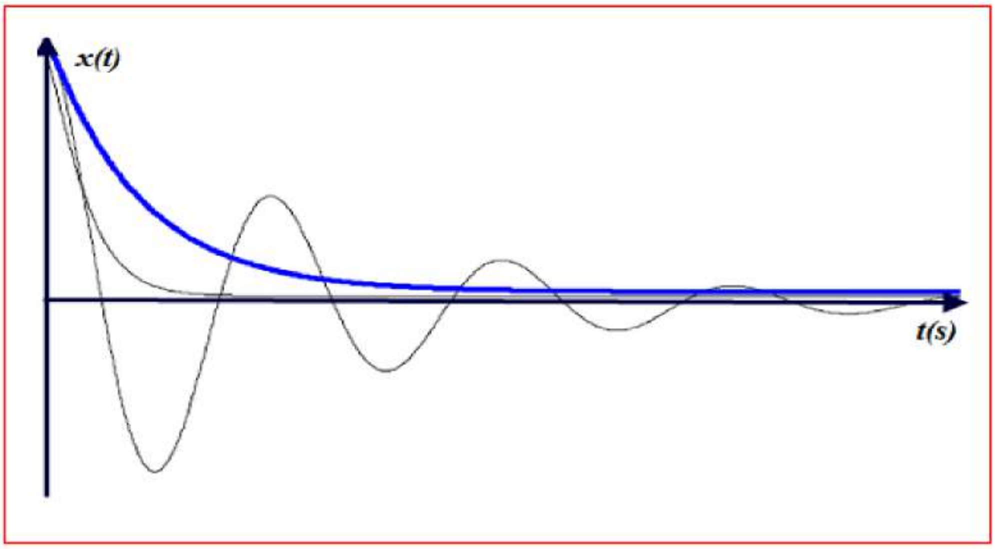
$$\begin{cases} r_{1=}{-\lambda} + \sqrt{\lambda^2 - w_0^2} \\ r_{2=}{-\lambda} - \sqrt{\lambda^2 - w_0^2} \end{cases}$$

The general solution $x(t)$ is written as a linear combination of the two solutions corresponding to each of the values of r:

$$x(t) = e^{-\lambda t}(C_1 e^{\sqrt{\lambda^2 - w_0^2}\, t} + C_2 e^{-\sqrt{\lambda^2 - w_0^2}\, t})$$

Noticed :

- Constants C_1 and C_2 can be determined from two initial conditions.
- Since the values r_1 and r_2 are negative, the solution $x(t) \to 0$ when $t \to \infty$.
- This type of movement is called **aperiodic damped movement** or **strongly damped (over-damped) movement.**



In blue, aperiodic damped movement (strong damping, over-damped)

4.2. second case: over-damped or aperiodic system (Critical damping):

$$\Delta = 0 \Rightarrow (\lambda^2 - w_0^2) = 0 \Rightarrow \lambda = w_0 \text{ ou } \xi = 1$$

In this case, the characteristic equation (4) has a double root:

$$r_1 = r_2 = \lambda$$

The general solution $x(t)$ is written in the form: $x(t) = C(t)e^{-rt}$ ... (4) and we obtain:

$$x(t) = (C_1 + C_2 t)\, e^{-\lambda t}$$

Demonstration:

We change the variable: $y(t) = \dot{x} + \lambda x \Rightarrow \frac{dy(t)}{dt} = \ddot{x} + \lambda \dot{x}$

The differential equation of motion $\ddot{x} + 2\lambda\dot{x} + w_0^2 x = 0$ becomes:

$\frac{dy(t)}{dt} + \lambda y(t) = 0$...(5) because:

$\frac{dy(t)}{dt} + \lambda y(t) = \ddot{x} + \lambda \dot{x} + \lambda(\dot{x} + \lambda x) = \ddot{x} + 2\lambda\dot{x} + w_0^2 x = 0$ with $(\lambda^2 = w_0^2)$

So to solve the differential equation of motion $\ddot{x} + 2\lambda\dot{x} + w_0^2 x = 0$,

we are looking for solutions to the following system of equations:

$$\begin{cases} \frac{dy(t)}{dt} + \lambda y(t) = 0 \ldots (a) \\ y(t) = \dot{x} + \lambda x \quad \ldots (b) \end{cases}$$

From equation (a): $\frac{dy(t)}{dt} + \lambda y(t) = 0 \Rightarrow \frac{dy(t)}{y(t)} = -\lambda dt$; we find : $y(t) = C_2 e^{-\lambda t}$.

From equation (b):

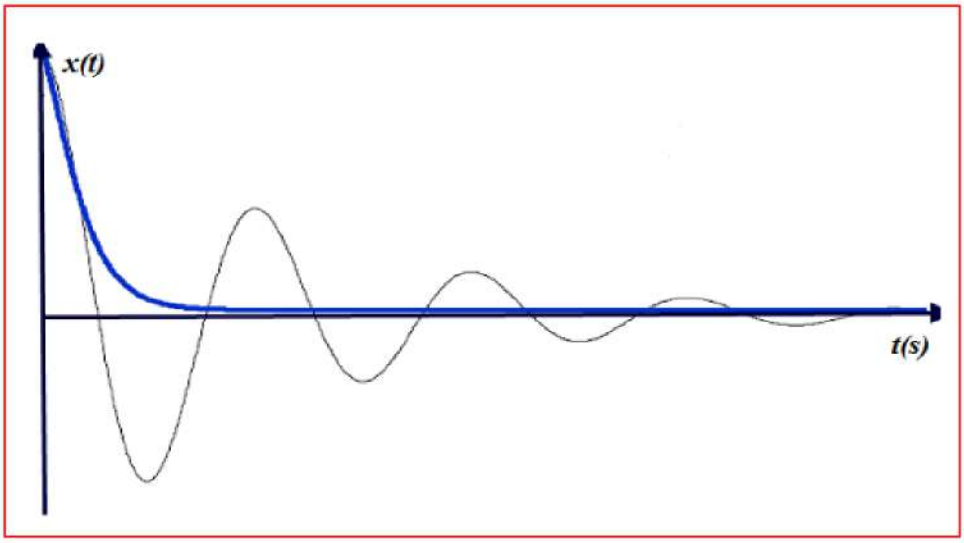
$y(t) = \dot{x} + \lambda x \Rightarrow C_2 e^{-\lambda t} = \dot{C}(t)e^{-\lambda t} + C(t)(-\lambda)e^{-\lambda t} + \lambda C(t)e^{-\lambda t}$;We obtain: $\dot{C}(t) = C_2$therefore $\frac{dC(t)}{dt} = C_2$. From where $C(t) = C_1 + C_2 t \ldots (5)$.

We replace equation (5) in (4), we obtain the general solution:

$$x(t) = (C_1 + C_2 t)\, e^{-\lambda t}$$

Noticed :

- Constants C_1 and C_2 can be determined from two initial conditions.
- the movement is also aperiodic; this state of the system **is called critical state** or **aperiodic damped movement (critical damped movement).**
- In this state $x(t)$ **tends to *0* faster than strong damping.**
- The system returns to its equilibrium position as quickly as possible



In blue aperiodic damped movement (critical damped movement)

4.3. third case: under-damped or pseudoperiodic system (low damping):

$$\Delta < 0 \Rightarrow (\lambda^2 - w_0^2) < 0 \Rightarrow \lambda < w_0 \text{ ou } 0 < \xi < 1$$

In this case, the characteristic equation (4) has **two complex roots** :

Because :

$$\sqrt{\Delta} = \sqrt{\lambda^2 - w_0^2} = \sqrt{(-1)(w_0^2 - \lambda^2)} = \sqrt{(i^2)(w_0^2 - \lambda^2)} = i\sqrt{(w_0^2 - \lambda^2)} = i\,w'$$

$$\begin{cases} r_{1=}-\lambda + i\sqrt{(w_0^2 - \lambda^2)} = -\lambda + i\,w' \\ r_{2=}-\lambda - i\sqrt{(w_0^2 - \lambda^2)} = -\lambda - i\,w' \end{cases}$$

With : $w' = \sqrt{(w_0^2 - \lambda^2)}$.

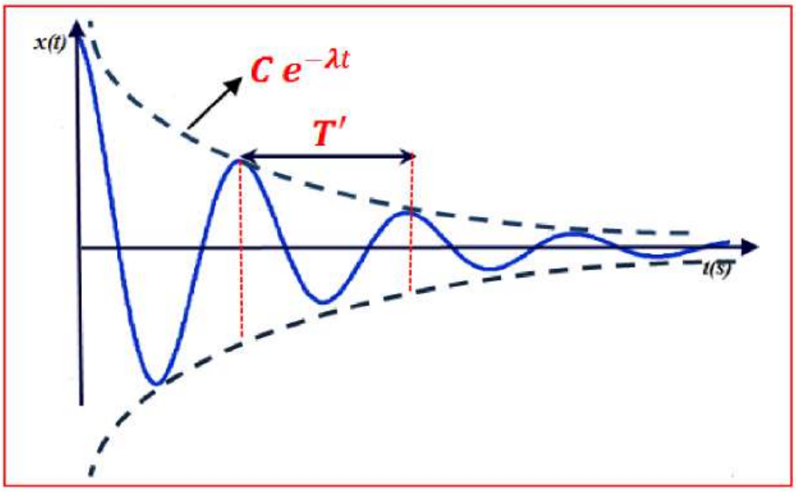
w': Is **the pseudo-pulsation, and**T': $\frac{2\pi}{w'}$

The general solution $x(t)$ is written as a linear combination of the two solutions corresponding to each of the values of r:

$$x(t) = e^{-\lambda t}(C_1 e^{i w' t} + C_2 e^{-i w' t}) \text{ or } x(t) = C\,e^{-\lambda t}\cos(w't + \varphi)$$

Noticed :

- The constants C and φcan be determined from two initial conditions.
- The amplitude$C\,e^{-\lambda t}$ and as a function of time, it tends towards 0 when time tends towards ∞ which implies the solution $x(t)\to 0$ when $t\to\infty$.
- the movement has a certain regularity, because the time T' which elapses between two successive maxima is constant.
- the movement is under-damped and this type of movement is called **pseudoperiodic movement** or **weakly damped movement.**
- **the pseudo-period**T': $\frac{2\pi}{w'}$ is constant.



Pseudo periodic movement (low damping):

4.3.1. Determination of the constants of C *and* φ :

We determine C *and* φ according to $x_0, \dot{x}_0; w_0$ and $\lambda$from the initial conditions:

$$\begin{cases} x(t) = C\, e^{-\lambda t}\cos(w't+\varphi) \\ \dot{x}(t) = -C\,\lambda\, e^{-\lambda t}\cos(w't+\varphi) - C\; w' e^{-\lambda t}\sin(w't+\varphi) \end{cases}$$

HAS $: t = 0 \Rightarrow \begin{cases} x(0) = x_0 \\ \dot{x}(0) = \dot{x}_0 \end{cases} \Rightarrow \begin{cases} x(0) = C\cos\varphi = x_0 & \dots(1) \\ \dot{x}(0) = -C\,\lambda\, cos(\varphi) - Cw' sin(\varphi) = \dot{x}_0 & \dots(2) \end{cases}$

And we have: $\cos^2(\varphi) + \sin^2(\varphi) = 1 \Rightarrow \sin(\varphi) = \sqrt{1-\cos^2(\varphi)} \qquad \dots(3)$

Equation *(3)* in *(2),* we obtain :

$$\begin{cases} (1) \Rightarrow \cos\varphi = {x_0}/{C} & \dots(4) \\ (2) \Rightarrow -C\,\lambda\cos(\varphi) - Cw'\sqrt{1-\cos^2(\varphi)} = \dot{x}_0 & \dots(5) \end{cases}$$

Equation *(4)* in *(5),* we obtain :

$$-C\,\lambda\,\frac{x_0}{C} - \sqrt{C^2}w'\sqrt{1-(\frac{x_0}{C})^2} = \dot{x}_0$$

$$-\lambda\, x_0 - w'\sqrt{C^2 - {x_0}^2} = \dot{x}_0 \quad \Rightarrow \sqrt{C^2 - {x_0}^2} = \frac{\dot{x}_0 + \lambda\, x_0}{w'}$$

We obtain : $C = \sqrt{\left(\frac{\lambda\, x_0}{w'} + \frac{\dot{x}_0}{w'}\right)^2 + {x_0}^2}$

And from the equation:

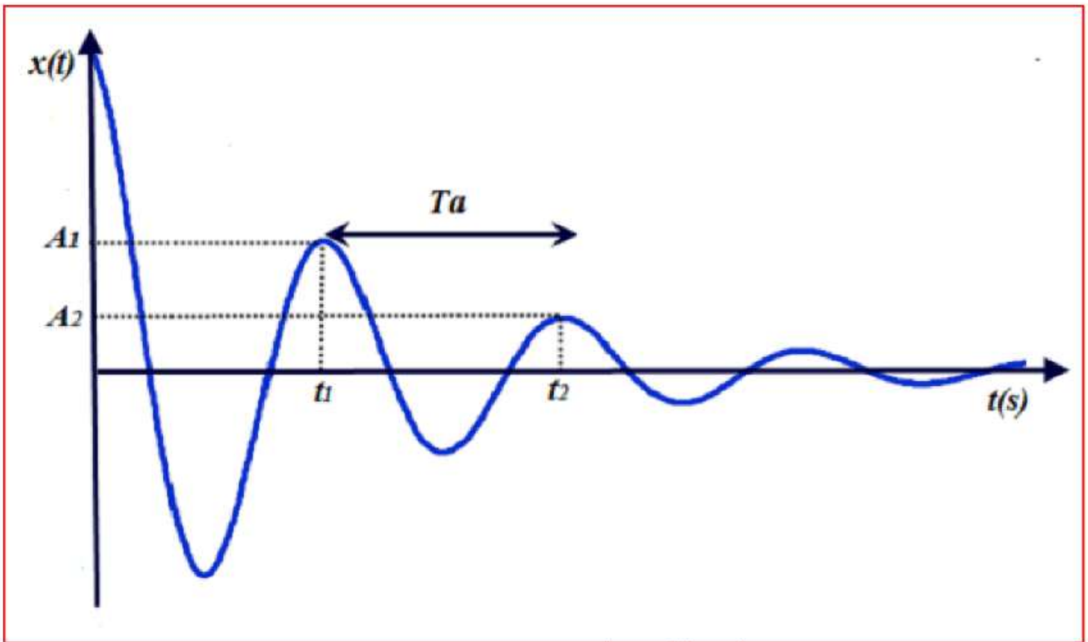
$$\frac{(2)}{(1)} \Rightarrow \frac{-C\,\lambda\cos(\varphi) - Cw'\sin(\varphi)}{C\cos\varphi} = \frac{\dot{x}_0}{x_0}$$

We obtain :

$$tg(\varphi) = \frac{-1}{w'}\left(\frac{\dot{x}_0}{x_0} + \lambda\right)$$

4.3.2. Logarithmic decrement:

Definition : we define the logarithmic decrement D_{ln} which measures the decrease in amplitude during a single period.



$$D_{ln} = ln\left[\frac{x(t)}{x(t+T')}\right] = \lambda\, T'$$

Demonstration:

$$D_{ln} = ln\left[\frac{x(t)}{x(t+T')}\right] = ln\left[\frac{C\, e^{-\lambda t}\cos(w't+\varphi)}{C\, e^{-\lambda(t+T')}\cos(w'(t+T')+\varphi)}\right] = ln\left[\frac{1}{e^{-\lambda T'}}\right] = ln\left[\frac{1}{e^{-\lambda T'}}\right]$$
$$= ln\left[e^{\lambda T'}\right] = \lambda\, T'$$

With **T '** is the pseudoperiod of the system.

After ***n*** pseudoperiod, the logarithmic decrement is given by:

$$D_{ln} = \frac{1}{n}\, ln\left[\frac{x(t)}{x(t+nT')}\right] = \lambda\, T'$$

We can determine the pseudoperiod T' according to the logarithmic decrement D_{ln} **:**

With :

$$\begin{cases} w'^2 = w_0^2 - \lambda^2 \\ D_{ln} = \lambda T' \Rightarrow \lambda = \dfrac{T'}{D_{ln}} \end{cases} \Rightarrow \left\{ (\frac{2\pi}{T'})^2 = (\frac{2\pi}{T_0})^2 - (\frac{T'}{D_{ln}})^2 \right.$$

We obtain : $T' = T_0\sqrt{1 + (\frac{D_{ln}}{2\pi})^2}$

Because :

w': Is the pseudo-pulsation. $w' = \frac{2\pi}{T'}$

w_0: Is the own pulsation. $w_0 = \frac{2\pi}{T_0}$

4.3.3. Quality factor (Surge factor):

The quality factor Q is the ratio between the mechanical energy and the energy loss during a pseudo period.

$$Q = 2\pi \frac{E(t)}{E(t) - E(t + T')} = \frac{2\pi}{1 - e^{-2\lambda T'}} \dots (1)$$

In the case of low depreciation:

$\lambda << w_0$ and $w' = \sqrt{w_0^2 - \lambda^2}$ so $w' \approx w_0$

$$e^{\varepsilon} = 1 + \varepsilon \Rightarrow e^{-2\lambda T'} = e^{-2\lambda(\frac{2\pi}{w_0})} \approx 1 - 4\pi \frac{\lambda}{w_0} \dots (2)$$

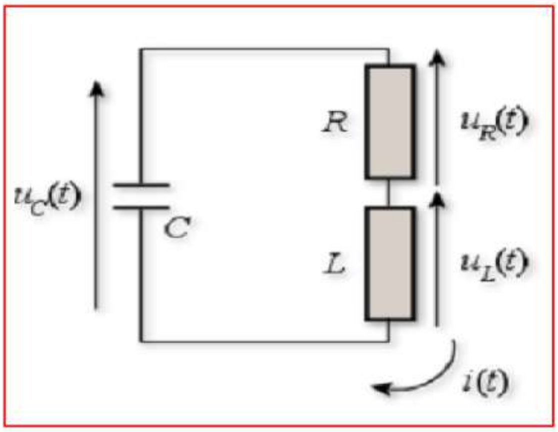
We replace equation ***(2)*** in ***(1),*** we find the simplified quality factor equation in the case of low depreciation:

$$Q = \frac{w_0}{2\lambda}$$

Noticed : there is an inverse proportionality between the quality factor and the damping, the lower the damping, the greater the quality factor of the system **.**

4. ***Electrical system:***

Or an electrical circuit, made up of 3 basic elements placed in series:



- a resistor of resistance ***R.***
- a capacitor of capacity ***C.***
- and a coil of inductance ***L.***

According to de Kirchhoff's law:

$$U_R + U_C + U_L = 0 \ldots(1)$$

With **:**

- $U_R = R\, i(t)\ldots(2)$
- $U_C = \frac{1}{C}\, q \quad \ldots(3)$
- $U_L = L\frac{di}{dt} \quad \ldots(4)$

And we have:

$$i(t) = \frac{dq}{dt} = \dot{q} \quad \Rightarrow \frac{di}{dt} = \frac{d^2q}{d^2t} = \ddot{q} \ldots(5)$$

We replace equations ***(2)*** , ***(3)*** , ***(4)*** and ***(5)*** in equation ***(1),*** we obtain:

$$R\, i(t) + \frac{1}{C}\, q + L\frac{di}{dt} = 0 \quad \Rightarrow \; R\,\frac{dq}{dt} + \frac{1}{C}\, q + L\,\frac{d^2q}{d^2t} = 0$$

$$R\,\dot{q} + \frac{1}{C}\, q + L\ddot{q} = 0 \quad \Rightarrow \; L\ddot{q} + R\,\dot{q} + \frac{1}{C}\, q = 0$$

$$\ddot{q} + \frac{R}{L}\,\dot{q} + \frac{1}{LC}\, q = 0 \quad \Rightarrow \; \ddot{q} + 2\lambda\,\dot{q} + w_0^2\, q = 0$$

$$\textbf{With :} \begin{cases} 2\lambda = \frac{R}{L} \\ w_0^2 = \frac{1}{LC} \end{cases} \Rightarrow \begin{cases} \lambda = \frac{R}{2L} \\ w_0 = \sqrt{\frac{1}{LC}} \end{cases}$$

Note: For critical damping :

$$\lambda = w_0 \quad \Rightarrow \quad \frac{R}{2L} = \sqrt{\frac{1}{LC}} \;\; ; \; R = R_C = 2\sqrt{\frac{L}{C}}$$

R_C **:** is the critical resistance.

Problem corrected

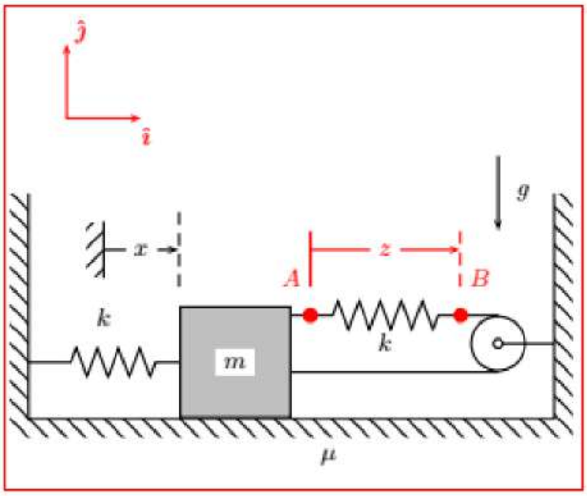
Problem 1:

The block shown to the right rests on a rough surface with coefficient of friction μ and

$m = 6$ kg, $\quad k = 128$ N/m.

a) If the block is displaced 3 cm to the right and released, for what values of μ will the block remain in that position?

b) With $\mu = 0.50$, if the block is displaced 30 cm to the right and released from rest, how long will it take the block to come to rest?



Solution :

In addition to the variable x identified in the problem statement, we also define z to be the stretch in the spring parallel with the cable system. As a one degree-of-freedom system, the variables x and z are directly related. The relative velocity across the spring can be identified as

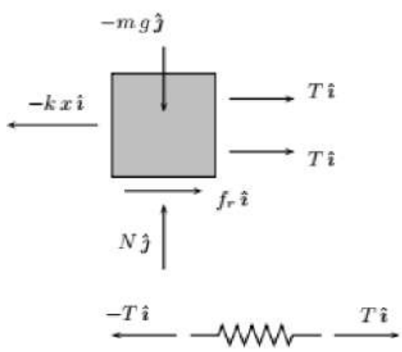
$$\begin{aligned} {}^{\mathcal{F}}\mathbf{v}_B - {}^{\mathcal{F}}\mathbf{v}_A &= \dot{z}\,\hat{\imath}, \\ &= (-\dot{x}\,\hat{\imath}) - (\dot{x}\,\hat{\imath}), \end{aligned}$$

so that $\dot{z} = -2\,\dot{x}$. Therefore the kinematic relationship becomes

$$z = -2\,x.$$

An appropriate free-body diagram for this system is shown to the right. Note that the unknown friction force is denoted as $f_r\,\hat{\imath}$ and the tension in the cable is T. Finally, examining the spring in the cable, the tension T and the displacement z are related as

$$T = k\,z = -2\,k\,x.$$



Applying linear momentum balance to the block yields

$$\sum \boldsymbol{F} = (2\,T - k\,x + f_r)\,\hat{\imath} + (N - m\,g)\,\hat{\jmath} = m\,\ddot{x}\,\hat{\imath} = m\,{}^{\mathcal{F}}\mathbf{a}_G,$$

and in terms of x the equation of motion becomes

$$m\,\ddot{x} + 5\,k\,x = f_r.$$

If the block slips then $f_r = -\mu m g \, \mathrm{sgn}(\dot{x})$ while is sticking occurs $|f_r| \leq \mu m g$.

a) If the block is in static equilibrium at a displacement $x = x_{eq}$, then $\ddot{x}_{eq} \equiv 0$ and the equation of motion reduces to

$$5 k x_{eq} = f_r,$$

so that equilibrium is maintained provided

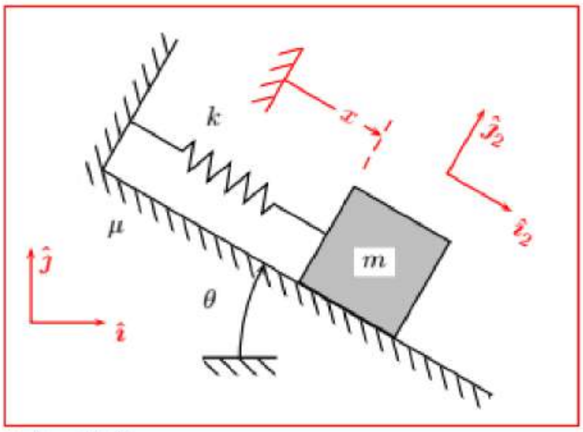
$$|f_r| = |5 k x_{eq}| \leq \mu m g.$$

This inequality is satisfied provided

$$|x_{eq}| \leq \frac{\mu m g}{5 k}.$$

Problem 2:

The system shown in the fig- ure has mass m and rests on a plane inclined at an angle φ. The coefficient of friction for the rough surface is μ and the system is re- leased from rest at the unstretched position of the spring (with stiffness k).



a) If $\mu = 0$, what is the equilibrium dis- placement of the mass (as measured from the unstretched position)?

b) For $\mu > 0$, at what angle θ does the block begin to slip?

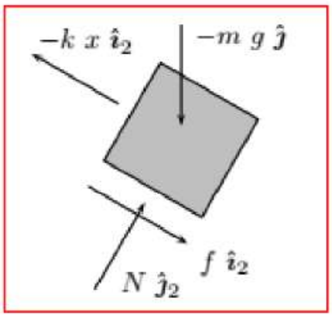
c) Find the value of θ so that the system comes to rest after one full cycle *exactly* at the equilibrium position of the sys- tem found in part a (so that the friction force vanishes when the system comes to rest), with:

$m = 2$ kg, $k = 32$ N/m, $\mu = 0.35$.

Solution :

The unit directions $\hat{\imath}_2$ and $\hat{j}_2$ are defined to be coincident with the inclined plane and the coordinate x represents the displacement of the mass from the unstretched position of the spring, as shown in the figure.

A free-body diagram for this system is shown to the right. Notice that the force in the upper spring depends on z, rather than x, while the friction force has an unknown magnitude f. Because the disk is assumed to roll without slip, we are unable to specify the value of f, but instead can relate the dis- placement and rotation of the disk through the coordinate relations above.



Therefore, linear momentum balance yields the following equations:

$$\begin{aligned}
\sum \boldsymbol{F} &= m\, {}^{\mathcal{F}}\mathbf{a}_G, \\
\left(f - k\, x\right) \hat{\imath}_2 + \left(N\right) \hat{\jmath}_2 - \left(m\, g\right) \hat{\jmath} &= m\, \ddot{x}\, \hat{\imath}_2, \\
\left(f - k\, x + m\, g\, \sin\theta\right) \hat{\imath}_2 + \left(N - m\, g\, \cos\theta\right) \hat{\jmath}_2 &= m\, \ddot{x}\, \hat{\imath}_2.
\end{aligned}$$

Therefore, this leads to the following scalar equations in the $\hat{\imath}_2$ and $\hat{\jmath}_2$ directions:

$$\begin{aligned}
m\, \ddot{x} + k\, x &= f + m\, g\, \sin\theta, \\
N &= m\, g\, \cos\theta.
\end{aligned}$$

a) If $\mu = 0$, then the friction forces vanishes and the first of the above equations reduces

to:

$$m\, \ddot{x} + k\, x = m\, g\, \sin\theta.$$

The equilibrium displacement of the mass, x_{eq}, then can be found to be:

$$x_{\text{eq}} = \frac{m\, g}{k}\, \sin\theta.$$

b) With $\mu \neq 0$, an equilibrium state is maintained provided:

$$\begin{aligned}
\left|f\right| &\leq \mu\, N, \\
\left|k\, x - m\, g\, \sin\theta\right| &\leq \mu\, m\, g\, \cos\theta.
\end{aligned}$$

Therefore, if the system is released from $x = 0$, the block begins to slide when:

$$\tan\theta = \mu.$$

c) Define z to be a new coordinate measuring the displacement of the system from static equilibrium:

$$z = x - \frac{m\, g}{k}\, \sin\theta,$$

so that the equations of motion become:

$$m\, \ddot{z} + k\, z = f, \qquad N = m\, g\, \cos\theta.$$

with initial displacement $z(0) = -(m\, g\, \sin\theta)/k$. Over one complete cycle of motion, for a frictionally damped system the amplitude decreases by a value:

$$\Delta\, A = -\frac{4\, \mu\, N}{k}.$$

Therefore, if the system comes to rest at exactly the equilibrium position, then this decrease in amplitude must exactly match the initial displacement. That is:

$$|\Delta\, A| = \left|-\frac{4\, \mu\, N}{k}\right| = \left|-\frac{m\, g}{k}\, \sin\theta\right| = |z(0)|.$$

Solving for θ:

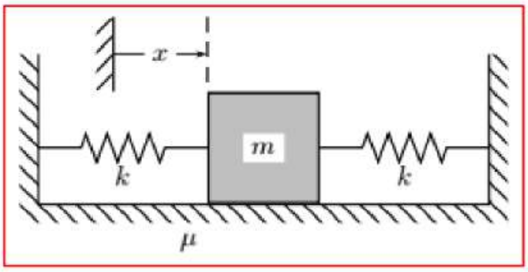
$$\tan\theta = 4\, \mu.$$

Problem 3:

For the spring-mass system with Coulomb damping:

a) determine the governing equations of motion;

b) what is the period of each oscillation?



Solution

a) We measure the displacement of the mass from the static equilibrium of the frictionless system, i.e., $\mu = 0$, so that the acceleration of the block is ${}^{\mathcal{F}}\mathbf{a}_G = \ddot{x}\hat{\imath}$. Thus linear momentum balance yields:

$$m\ddot{x}\hat{\imath} = F_{\text{spring}}\hat{\imath} + f_\mu\hat{\imath} + (N - mg)\hat{\jmath}.$$

The spring force is $F_{\text{spring}} = -2kx$, while the force due to sliding friction opposes the velocity and is simply:

$$f_\mu = -\mu mg\frac{\dot{x}}{|\dot{x}|},$$

since the normal force balances the gravitational force, i.e., $N = mg$. If the block is stationary the magnitude of the frictional force is less than μmg. Therefore, the governing equations of motion are:

$$m\ddot{x} + 2kx = f_\mu, \qquad \text{with} \quad \begin{array}{rcll} f_\mu & = & -\mu mg\frac{\dot{x}}{|\dot{x}|} & |\dot{x}| \neq 0, \\ |f_\mu| & \leq & \mu mg & \dot{x} = 0. \end{array}$$

b) Coulombic damping does not effect the frequency of oscillation, which is simply:

$$\omega = \sqrt{\frac{2k}{m}}.$$

Therefore the period of the oscillation is:

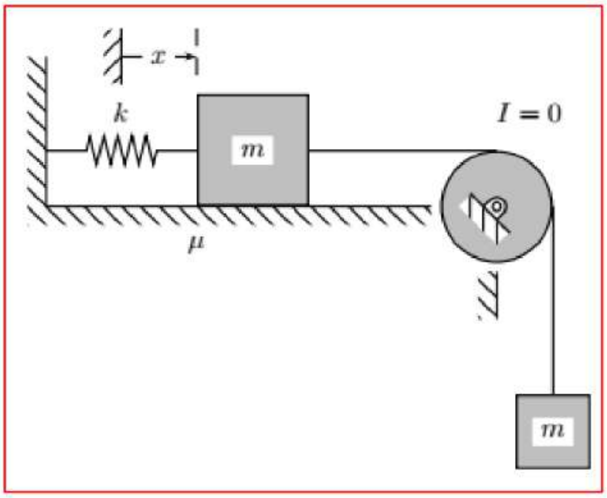
$$T = \frac{2\pi}{\omega} = \frac{2\pi\sqrt{m}}{\sqrt{2k}}.$$

Problem 4:

For the spring-mass system with Coulombic damping, x is measured from the unstretched position of the spring. If the coefficient of friction is μ and the gravitational constant is g:

a) determine the governing equations of motion;

b) if the system is released from rest in the unstretched position ($x(0) = 0$, $\dot{x}(0) = 0$), for what values of μ will the system move;



c) what is the displacement (from the un- stretched position) of the upper block when it first comes to rest?

Solution

Notice that x describes the displacement of both masses and, since the pulley is massless, the tension in the string connecting the masses is constant, say T. Also, notice that in part c we ask for the displacement from the unstretched position of the spring, rather than from equilibrium. Therefore we include the gravitational force which will influence this result.

a) With the frictional force defined as $\boldsymbol{F} = f\ \hat{\imath}$, linear momentum balance on the upper and lower block yields:

$$\begin{aligned} m\ddot{x} + kx &= f + T, \\ m\ddot{x} &= mg - T, \end{aligned}$$

where the frictional force is defined as:

$$\begin{aligned} f &= -\mu mg \tfrac{\dot{x}}{|\dot{x}|}, \quad \dot{x} \neq 0, \\ |f| &\leq \mu mg, \qquad\quad \dot{x} = 0. \end{aligned}$$

Eliminating the unknown tension T, the equation of motion is given as:

$$2m\ \ddot{x} + kx = f + mg,$$

where f is defined as above and depends on the motion of the system, that is, the value of $\dot{x}$.

b) If the system is released from rest in the unstretched position, it will remain there provided the magnitude of the frictional force is less than μmg—the transition to movement occurs when $|f| = \mu mg$. Thus the system does not move is $|f| \leq \mu mg$ *and* $\ddot{x} = 0$. Substituting these conditions into the equations of motion we find:

$$|f| = |kx - mg| \leq \mu mg,$$

which, solving for μ with $x(0) = 0$, implies that the system does not move if $\mu \geq 1$. Therefore, the system *does* move when:

$$\mu < 1.$$

c) The displacement of the upper block when it first comes to rest is:

$$x_1 = \frac{2(1-\mu)mg}{k}.$$

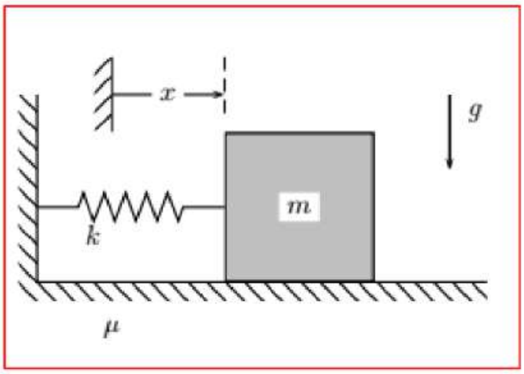
This can be found by either solving the equations of motion explicitly, or through a work-energy analysis. Since the initial and final kinetic energy is zero, the work done by the frictional force balances out the change in potential energy from the spring and gravity.

Problem 5:

For the spring-mass system with Coulombic damping, x is measured from the unstretched position of the spring. If the coefficient of friction is μ and the gravitational constant is g:

a) determine the governing equations of motion;

b) if the system is released from rest, so that $\dot{x}(0) = 0$, for what range of initial displacements (from the unstretched position) will the block come to rest when the block first comes again to rest ($\dot{x}(t_1) = 0$ for $t_1 > 0$)?



Solution

a) The equations of motion can be written as:

$$m\ddot{x} + kx = f,$$

where f is the force due to friction, modeled by Coulomb's law of friction as:

$$f = -\mu mg\frac{\dot{x}}{|\dot{x}|}, \quad \dot{x} \neq 0,$$

$$|f| \leq \mu mg, \quad \dot{x} = 0.$$

b) If the system is released from rest, the initial displacement must be sufficiently large so that the block slides, rather than remaining at rest. Sliding does not occur if the force due to friction is sufficient to balance the elastic force, that is, $\mu mg \geq f = kx(0)$. Thus, solving for $x(0)$ we find, that for sliding to occur:

$$|x(0)| > \frac{\mu mg}{k}.$$

However, if $|x(0)|$ is too large, the system will undergo multiple reversals as the amplitude of the motion decays. Consider the block sliding to the left ($\dot{x} < 0$), released from rest with initial displacement $x(0) > \frac{\mu mg}{k}$. Thus the equation of motion becomes:

$$m\ddot{x} + kx = \mu mg,$$

which has the general solution:

$$x(t) = \left[x(0) - \frac{\mu mg}{k}\right]\cos\left(\frac{k}{m}t\right) + \frac{\mu mg}{k}.$$

Therefore, when the block comes again to rest at time t_1 (unknown), the mass is at the position:

$$x(t_1) = 2\frac{\mu mg}{k} - x(0).$$

At this point, the block sticks if and only if $|x(t_1)| \leq \frac{\mu mg}{k}$. Therefore, solving for $x(0)$, we find that:

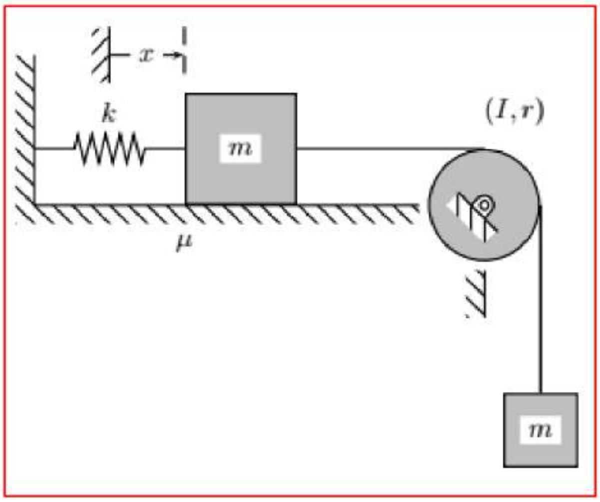
$$x(0) \leq 3\frac{\mu mg}{k}.$$

So for a block with $x(0) > 0$, the allowable range for $x(0)$ is $\frac{\mu mg}{k} < x(0) \leq 3\frac{\mu mg}{k}$. Together with an identical argument for $x(0) < 0$ yields the total allowable range as:

$$\frac{\mu mg}{k} < |x(0)| \leq 3\frac{\mu mg}{k}.$$

Problem 6:

For the system shown to the right, x is mea- sured from the *unstretched position of the spring*. Each block has mass m and the disk has moment of inertia I and radius r. The coefficient of friction between the upper block and the table is μ. If the gravitational con- stant is g:



a) find the equations of motion which de- termine $x(t)$;

b) what is the minimum value of μ so that

c) the system slips when release from

d) rest with $x(0) = 0$;

e) what is the period of the free oscilla- tions?

f) if the system is released from rest, what is the range of initial displacements $x(0)$ so that

g) the systems comes to rest after exactly one complete cycle?

Solution

We begin by defining two additional coordinates, θ, which describes the rotation of the disk in the $-\hat{k}$ direction (clockwise), and y which measures the displacement of the hanging mass in the $-\hat{\jmath}$ direction. These additional coordinates are related to the displacement of the upper mass by the constraint equations:

$$y = x, \qquad \theta = \frac{x}{r}.$$

a) On each mass the equations of motion can be written as:

$$\sum \boldsymbol{F} = \left(-k\,x + T_1 + f\right)\hat{\imath} + \left(N - mg\right)\hat{\jmath} = m\ddot{x}\,\hat{\imath} = m\,{}^{\mathcal{F}}\mathbf{a}_{G_1},$$

$$\sum \boldsymbol{M}_O = \left(T_1 r - T_2 r\right)\hat{\boldsymbol{k}} = -I^O\,\ddot{\theta}\hat{\boldsymbol{k}} = I^O \boldsymbol{\alpha}_{\mathcal{D}/\mathcal{F}}\mathcal{F},$$

$$\sum \boldsymbol{F} = \left(T_2 - mg\right)\hat{\jmath} = -m\ddot{y}\,\hat{\imath} = m\,{}^{\mathcal{F}}\mathbf{a}_{G_2},$$

Notice that $I^O = I \neq 0$, so that provided $\ddot{\theta} \neq 0$ the tensions T_1 and T_2 are not equal. Taking the components of these equations and eliminating the unknowns (T_1, T_2), while using the constraint equations, we find that the equation of motion for this system reduces to:

$$\left(2m + \frac{I}{r^2}\right)\ddot{x} + k\,x = f + mg,$$

where f the value of the frictional force in the $\hat{\imath}$ direction, can be written as:

$$f = \begin{cases} -\mu m g \frac{\dot{x}}{|\dot{x}|}, & \dot{x} \neq 0 \end{cases}$$

b) The minimum value for slip is simply $\mu_{\min} = 0$. If we would like to find the range of μ for which slip occurs, we resort to the value of f at static equilibrium. Assuming $(\dot{x}, \ddot{x}) = (0, 0)$, the equations of motion reduce to:

$$kx = f_{\text{static}} + mg,$$

where f_{static} represents the force required to maintain static equilibrium. Solving for this quantity and using the frictional inequality, we find:

$$|f_{\text{static}}| = |kx_0 - mg| \leq \mu mg.$$

Therefore, solving for μ yields:

$$\mu \geq \left|\frac{kx_0}{mg} - 1\right| = \left|1 - \frac{kx_0}{mg}\right|,$$

which provides a necessary condition for *sticking* at $x = x_0$. So for sliding to occur for $x_0 = 0$, this implies that $\mu < 1$.

c) The period of oscillation for a frictionally damped system is identical to that of an undamped system. Therefore:

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{2m + \frac{I}{r^2}}{k}}.$$

d) Let δ describe the displacement of the system from equilibrium. The amplitude of oscillation will decay by a value of $\Delta = 4\mu mg/k$ over one cycle of motion. Therefore, Since the system will come to rest within the range:

$$-\frac{\mu mg}{k} < |\delta_{\text{final}}| \leq \frac{\mu mg}{k},$$

the initial displacement δ_0 from the equilibrium in the absence of friction must be in the range:

$$-\frac{\mu mg}{k} + \Delta = \frac{3\mu mg}{k} < |\delta_0| \leq \frac{5\mu mg}{k} = \frac{\mu mg}{k} + \Delta.$$

However, the equilibrium position corresponds to $x_{\text{eq}} = \frac{mg}{k}$, and so the allowable range of x_0 is:

$$\frac{3\mu mg}{k} < \left|x_0 - \frac{mg}{k}\right| \leq \frac{5\mu mg}{k}.$$

Chapter 4

Oscillations forced to one degree of freedom.

1. Introduction :

It was seen that the damping of oscillations was due to a reduction in mechanical energy in the form of dissipated heat. To compensate for these energy losses and maintain (conserve) the oscillations, a source of energy is required through an external force. We will therefore add a force F(t) which varies as a function of time.

2. Equation of motion of a damped and forced system :

The Lagrange equation of a damped and forced system with one degree of freedom is given by:

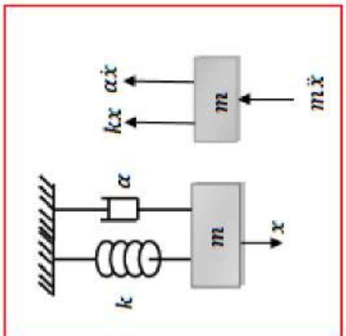
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)=-\frac{\partial D}{\partial \dot{q}}+F(t)$$

With :

$F(t)$: The external force, and a periodic function of time with a pulsation (Ω).

$F(t)$: $F_0 \cos(\Omega t)$.

Example : **- Horizontal elastic pendulum (mass-spring-damper):**



- The kinetic energy of the system: $T=\frac{1}{2}m\dot{x}^2$
- The potential energy of the system: $V=\frac{1}{2}kx^2$
- The dissipation function: $D=\frac{1}{2}\alpha\dot{x}^2$

- The Lagrange function:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\left(\frac{\partial L}{\partial x}\right)=-\frac{\partial D}{\partial \dot{x}}+F(t)\ \ldots(1)$$

Or : $L=T-V=\frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^2$

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=m\,\ddot{x}\\ \left(\frac{\partial L}{\partial x}\right)=-K\,x\ \ldots(2)\\ \frac{\partial D}{\partial \dot{x}}=\alpha\dot{x}\end{cases}$$

By replacing equation ***(2)*** in ***(1)*** We obtain:

$$m\ddot{x}+\alpha\dot{x}+k\,x=F_0\cos(\Omega t)$$

We then divide by m and we find:

$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{K}{m}x = \frac{F_0}{m}\cos(\Omega t)$, The form of the differential equation is:

$$\ddot{x} + 2\lambda\dot{x} + w_0^2 x = \frac{F_0}{m}\cos(\Omega t)$$

Or :

- λ: is **the damping factor** , with: $\lambda = \frac{\alpha}{2m}$.
- w_0: is **the system's own pulsation** , with **:** $w_0^2 = \frac{K}{m}$.
- Either $\xi = \frac{\lambda}{w_0}$ (unitless), is **the damping ratio.**

3. Solution of the equation of motion:

The differential equation of motion $\ddot{x} + 2\lambda\dot{x} + w_0^2\, x = \frac{F_0}{m}\cos(\Omega t)$ ... $(1)$is a second-order linear differential equation with constant coefficients with right hand side.
The general solution to this differential equation is the sum of two terms:

- A solution of the equation without a second member: homogeneous solution $x_H(t)$.
- A solution of the equation with right hand side: particular solution$x_p(t)$.

The total solution of the equation of motion will therefore be: $x(t) = x_H(t) + x_P(t)$.

3.1. Homogeneous solution (start of movement = transient regime):

The homogeneous solution corresponds to the solution of the differential equation without a second member

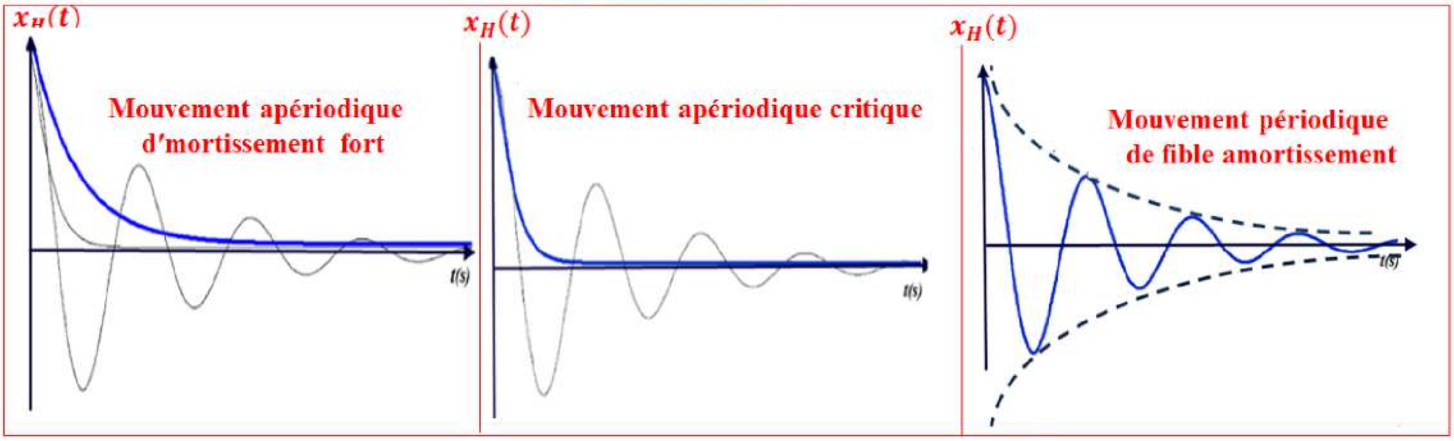
$$\ddot{x} + 2\lambda\dot{x} + w_0^2 x = 0$$

The expression for the homogeneous solution $x_H(t)$ depends on the Δ' , but in all cases $x_H(t)$ tends to zero after a certain time.

Noticed :

Whatever the nature of the movement: aperiodic with strong damping or aperiodic critical or periodic with weak damping, the homogeneous solution $x_H(t)$ of the equation$\ddot{x} + 2\lambda\dot{x} + w_0^2 x = 0$ tends to zero after a certain time; and we call this regime or $x_H(t)$ is different from zero: **The transitional regime .**

Régime transitoire



3.2. Special solution (permanent (forced) regime):

When the component $x_H(t)$ is negligible, at the end of the transitional regime . All that remains is **the particular solution** $x_P(t)$, which is the solution imposed by the excitation function (**the external force**). We say that we are in a **forced** or **permanent regime** .

The exciting force forces the mechanical system to follow a temporal evolution equivalent to its own. So if $F(t)$ is a periodic pulsation function (Ω) ; then the particular solution $x_P(t)$ will be a periodic function of the same pulsation (Ω) .

The oscillations of the body of **mass m** are not necessarily in phase with the **exciting force** and present a phase shift noted (φ). The particular solution corresponding to the **steady state** is written in the form:

$$x_P(t) = x_0 \cos(\Omega t + \varphi) \ldots (1)$$

(x_0): The amplitude of the vibration.
(φ): The phase shift between **the applied force** and the **vibration of the mass m** .

Determination of amplitude (x_0) and the phase shift (φ) :

To determine the amplitude (x_0) and the phase shift (φ) , we use **the complex number method** which allows us to transform the differential equation into an algebraic equation.

Note that : Z is a complex number, We can write **Z** in the following forms:
$Z = X + i\,Y$

$$Z = |Z|e^{i\theta}$$

$$Z = |Z|(\cos\theta + i\,\sin\theta)$$

With :

- $|Z|$ is the modulus of Z, $|Z| = \sqrt{X^2 + Y^2}$.
- θ is the argument of Z, $\theta = arctan(Y/X)$.

We have the external force is a periodic function of the pulsation time (Ω) :

$$F(t) = F_0 \cos(\Omega t) \dots (2)$$

So we can write equation (1) and (2) using the following complex notations:

$$F(t) = F_0 \cos(\Omega t) = Re\left[F_0 e^{i(\Omega t)}\right] \qquad \dots (3)$$

$$x_P(t) = x_0\, cos(\Omega t + \varphi) = x_0 e^{i(\Omega t + \varphi)} = x_0 e^{i(\varphi)} e^{i(\Omega t)} \dots (4)$$

We pose : $X_0 = x_0\, e^{i(\varphi)}$ therefore: $x_P(t) = X_0 e^{i(\Omega t)} \quad \dots (5)$

$$\dot{x}_P(t) = i\,\Omega\, X_0 e^{i(\Omega t)} \quad \dots (6)$$

$$\ddot{x}_P(t) = -\,\Omega^2\, X_0 e^{i(\Omega t)} \dots (7)$$

We replace equations *(3)* , *(5)* , *(6)* and *(7)* in the differential equation

$$\ddot{x} + 2\lambda\dot{x} + w_0^2\, x = \frac{F_0}{m} e^{i(\Omega t)}.$$

We obtain : $[-\,\Omega^2 + 2\lambda i\,\Omega + w_0^2]\, X_0 = \frac{F_0}{m}$

SO :

$X_0 = \dfrac{\frac{F_0}{m}}{[(w_0^2 - \Omega^2) + i\,2\lambda\,\Omega]}$ and, on the other hand $X_0 = x_0\, e^{i(\varphi)}$

We now identify the module $x_0 = |X_0|$ and the argument (φ) of this complex number:

$$\varphi = arg(X_0) = arg\left[\frac{\frac{F_0}{m}}{[(w_0^2 - \Omega^2) + i\,2\lambda\,\Omega]}\right] = arg\left[\frac{F_0}{m}\right] - arg\left[(w_0^2 - \Omega^2) + i\,2\lambda\,\Omega\right]$$

$$= 0 - arctan\left[\frac{2\lambda\,\Omega}{(w_0^2 - \Omega^2)}\right]$$

SO : $\varphi = -arctan\left[\frac{2\lambda\,\Omega}{(w_0^2 - \Omega^2)}\right]$;((φ): the phase shift)

$$x_0 = |X_0| = \left| \frac{\frac{F_0}{m}}{[(w_0^2 - \Omega^2) + i\,2\lambda\,\Omega]} \right| = \frac{\left|\frac{F_0}{m}\right|}{|(w_0^2 - \Omega^2) + i\,2\lambda\,\Omega|} = \frac{\frac{F_0}{m}}{\sqrt{(w_0^2 - \Omega^2)^2 + 4\lambda^2\,\Omega^2}}$$

$$= \frac{F_0}{m} \frac{1}{\sqrt{(w_0^2 - \Omega^2)^2 + 4\lambda^2\,\Omega^2}}$$

SO : $x_0 = \frac{F_0}{m} \frac{1}{\sqrt{(w_0^2 - \Omega^2)^2 + 4\lambda^2\,\Omega^2}}$;($x_0$: The amplitude of the vibration)

4. *Study of the steady state: resonance phenomenon:*

a- The amplitude of the vibration at resonance:

When the pulsation of the exciting force is varied Ω, the amplitude $x_0$reaches a maximum value when the derivative of $x_0$with respect to Ω is zero.

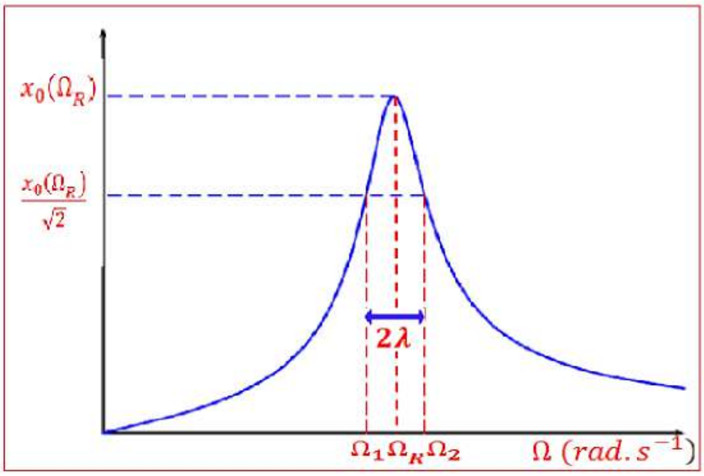
$$\frac{dx_0}{d\Omega} = -\frac{2\frac{F_0}{m}\Omega[\Omega^2 - w_0^2 + 2\lambda^2]}{[(w_0^2 - \Omega^2)^2 + 4\lambda^2\,\Omega^2]^{\frac{3}{2}}} = 0 \Rightarrow \Omega[\Omega^2 - w_0^2 + 2\lambda^2] = 0$$

$$\begin{cases} \Omega = 0 \\ \text{ou} \\ \Omega^2 - w_0^2 + 2\lambda^2 = 0 \end{cases}$$

There is a maximum at the resonance pulsation Ω_R, only if **the resonance condition** is verified: $w_0^2 > 2\lambda^2$.

$$\Omega_R = \sqrt{w_0^2 - 2\lambda^2}$$

We see in the figure that the amplitude x_0 is all the more important as the pulsation of the external force is close to the resonance pulsation Ω_R.



Noticed :

In the absence of sufficient damping, nothing would limit the amplitudes of the oscillations from amplifying, risking destruction of the system: the system enters into resonance. The consequences can be serious. We can cite two known cases:

- On April 18, 1850 in Angers, a regiment crossing at a rhythmic (harmonious) pace a suspension bridge spanning the Maine caused its destruction.
- On November 7, 1940, six months after its inauguration, the Tacoma suspension bridge (United States) (**following figure**) was destroyed by the effects of gusts of wind which, without being particularly violent (60 km.h $^{-1}$), were regular.

(a)

(b)

b- Quality factor and resonance:

Bandwidth :

The bandwidth, the band of pulsations for which the maximum amplitude is equal to: $x_0(\Omega_1) = x_0(\Omega_2) = \frac{x_0(\Omega_R)}{\sqrt{2}}$;

- the pulsations $(\Omega_1)$and $(\Omega_1)$are called **: the cut-off pulsations.**

➢ $(\Omega_2) - (\Omega_1) = 2\lambda$.

Quality factor:

As we saw in the main chapter, the quality factor is defined by the ratio of the specific pulsation to the width of the bandwidth

$$Q = \frac{w_0}{2\lambda}$$

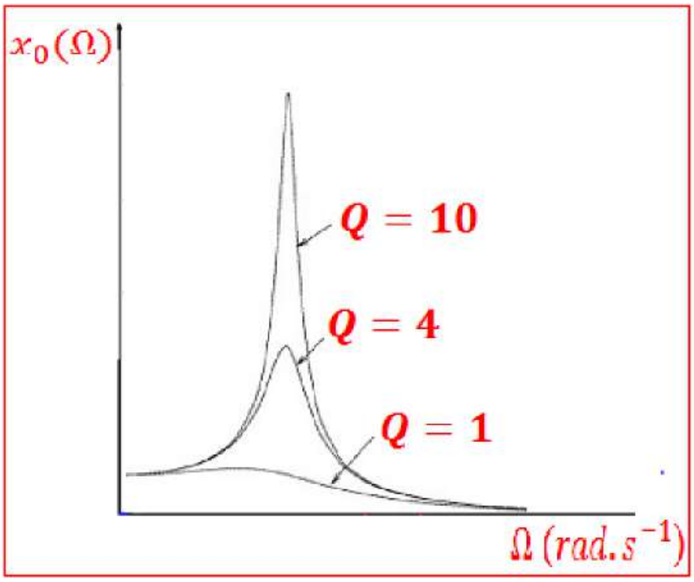
And for low depreciation: $\lambda << w_0 \Rightarrow \begin{cases} \Omega_R = \sqrt{w_0^2 - 2\lambda^2} = w_0 \\ (\Omega_2) - (\Omega_1) = 2\lambda \end{cases}$

So: we can write the quality factor in the form:

$$Q = \frac{\Omega_R}{(\Omega_2 - \Omega_1)}$$

Noticed :

To be able to reduce **the resonance amplitude** , we reduce the value of the **quality factor Q** , and for this we increase **the bandwidth, i.e.** we increase the damping factor and consequently the vibration amplitude also decreases (figure).



c- Phase variation depending on the pulsation :

We have already demonstrated the phase relationship (φ), which gives the phase shift between the external pulsation force $(\Omega)$and the vibration of our system

$$\varphi = -arctan\left[\frac{2\lambda\,\Omega}{(w_0^2 - \Omega^2)}\right] \Rightarrow tan(\varphi) = -\left[\frac{2\lambda\,\Omega}{(w_0^2 - \Omega^2)}\right] = \frac{-2\lambda\,\Omega}{w_0^2\left|1-\left(\frac{\Omega}{w_0}\right)^2\right|} = \frac{-\frac{2\lambda\,\Omega}{w_0^2}}{1-\left(\frac{\Omega}{w_0}\right)^2}$$

$$= \frac{-\frac{2\lambda}{w_0}\left(\frac{\Omega}{w_0}\right)}{1-\left(\frac{\Omega}{w_0}\right)^2}$$

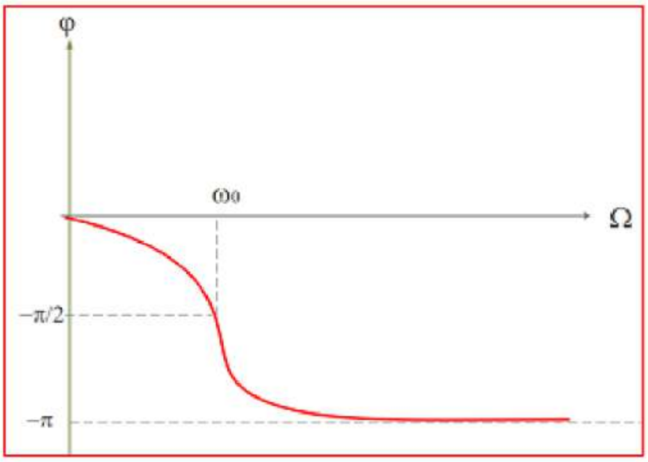
SO :

$$tan(\varphi) = \frac{-\frac{2\lambda}{w_0}\left(\frac{\Omega}{w_0}\right)}{1-\left(\frac{\Omega}{w_0}\right)^2}$$

- If $\Omega = 0 \Rightarrow tan(\varphi) = 0 \Rightarrow \varphi = 0$.
- If $\Omega = w_0 \Rightarrow tan(\varphi) = -\infty \Rightarrow \varphi = -\frac{\pi}{2}$.

Noticed :

- The oscillator is in **phase resonance** with the force applied to the system when $\varphi = -\frac{\pi}{2}$ For $\Omega = w_0 = \Omega_R$.
- The oscillator always lags in phase with respect to the force and this delay increases as the pulsation increases.



5. Mechanical impedance and electrical impedance:

5.1- Electrical impedance:

Electrical impedance is a quantity which characterizes the way in which the circuit slows down the flow of current by giving the ratio which exists between the source voltage and the resulting current.

When using the complex notation for electric current:

$$I = I_0 \, e^{i\Omega t}$$

The electrical impedance is defined by:

$$Z = \frac{V}{I} \quad (\,ohm\,)$$

- **Resistance :**

$$V_R = R\,I \Rightarrow Z_R = \frac{V_R}{I} = R$$

- **Coil :**

$$V_L = L\,\frac{dI}{dt} = \,i\Omega LI \Rightarrow Z_L = \frac{V_L}{I} = i\Omega L$$

- **Capacitor:**

  We have: $I = \frac{dq}{dt} \Rightarrow q = \int I\,dt$

  SO :

$$V_c = \frac{q}{C} = \frac{1}{C}\int I\,dt = \frac{I}{i\Omega C} \Rightarrow Z_C = \frac{V_C}{I} = \frac{1}{i\Omega C}.$$

We clearly observe **the phase shift** between **the current** and **the voltage** when we use the complex notation:

$$\begin{cases} Z_R = R = \,R\,e^{i0} \\ Z_L = i\Omega L = \Omega L\,e^{i\frac{\pi}{2}} \\ Z_C = \dfrac{1}{i\Omega C} = \dfrac{1}{\Omega C}e^{i(-\frac{\pi}{2})} \end{cases}$$

5.2- Mechanical impedance:

By analogy, we can also define the mechanical impedance, if we assume that the mechanical system subjected to a sinusoidal force of the form: $\mathbf{F(t)} = \boldsymbol{F_0} \mathbf{cos}(\Omega \boldsymbol{t})$, the point of application of this force moves with a speed $\mathbf{v(t)} = \boldsymbol{V_0} \mathbf{cos}(\Omega \boldsymbol{t} + \boldsymbol{\varphi})$; Mechanical impedance is a quantity that characterizes the way in which the system slows down the movement of the point of application, giving the relationship that exists between the complex amplitude of the force $\mathbf{F(t)}$and the speed of the point of application $\mathbf{v(t)}$.

$$Z = \frac{\mathbf{F(t)}}{\mathbf{v(t)}}$$

- **Shock absorber :**

The damping force is $\mathbf{F} = \boldsymbol{\alpha}\mathbf{v}$, we deduce the complex impedance of a shock absorber

$$Z_\alpha = \frac{\alpha v}{v} = \alpha$$

- **Mass :**

  According to Newton's 2nd law, the force F is written:

$$Z_m = \frac{m\ddot{a}}{v} = \frac{m\dot{v}}{v} = \frac{m\dot{v}}{v}$$

  With :

$$x(t) = X_0 e^{i(\Omega t)} \Rightarrow \dot{x}(t) = i\,\Omega\, X_0 e^{i(\Omega t)} \Rightarrow \ddot{x}(t) = \dot{v}(t) = i\Omega(i\Omega X_0 e^{i(\Omega t)}) = i\,\Omega\, v(t)$$

  So we obtain:

$$Z_m = \frac{m\, i\, \Omega\, v}{v} = im\ \Omega$$

- **Spring :**

  The restoring force of the spring is $F = K\,x$

$$Z_K = \frac{K\,x}{v} = \frac{K\,x}{\dot{x}} \Rightarrow$$

  With :

$$x(t) = X_0 e^{i(\Omega t)} \Rightarrow \dot{x}(t) = i\,\Omega\, X_0 e^{i(\Omega t)}$$

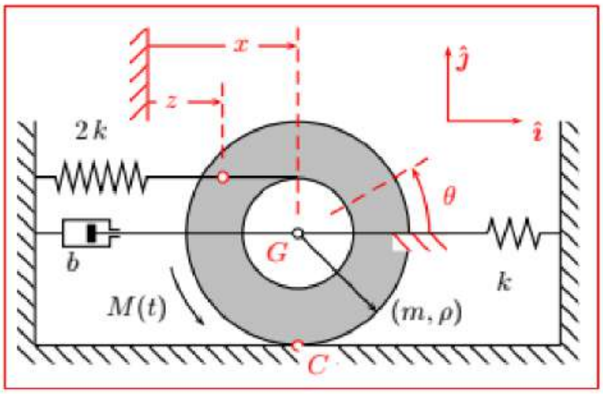
  We obtain :

$$Z_K = \frac{k}{i\Omega}$$

Problem corrected

Problem: 01

For the system shown to the right, the disk of mass m rolls without slip and the inner hub has radius $\rho/2$.



a) Find the equations of motion (in terms of the given parameters—*do not substi- tute in numerical values yet*);

b) If the applied moment takes the form:

$M(t) = (2 \text{ N} \cdot \text{m}) \sin(4t),$

find the steady-state amplitude of the translation of the center of the disk when:

$k = 16$ N/m, $\quad b = 2$ N/(m/s),

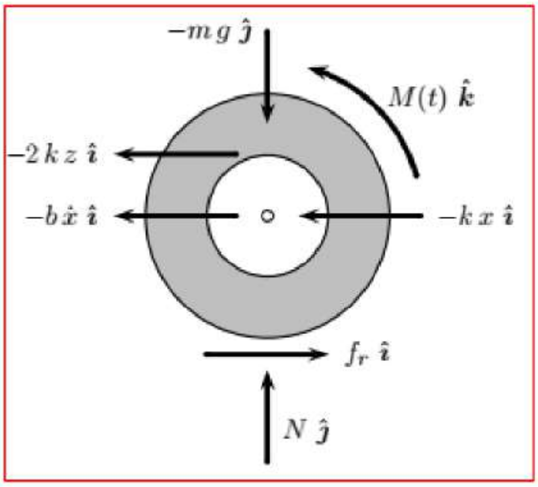
$m = 2$ kg, $\quad \rho = 0.125$ m

c) Determine the steady state amplitude of the friction force.

Solution

a) We identify the three coordinates x, z, and θ as shown in the figure above. These are related as:

$$x = -\rho\theta, \qquad z = \frac{3}{2}x.$$



An appropriate free-body diagram for this system is shown to the right. Since the disk is assumed to roll without slip, the equation of motion can be directly obtained with angular momentum balance about the contact point C

$$\sum \mathbf{M}_C = I^C \, \boldsymbol{\alpha}_{D/\mathcal{F}},$$

which yields

$$\left(\rho\,(k\,x+b\,\dot{x})+\frac{3\,\rho}{2}(2\,k\,z)+M(t)\right)\;\hat{\boldsymbol{k}}=\frac{3\,m\,\rho^2}{2}\;\ddot{\theta}\;\hat{\boldsymbol{k}}.$$

Using the above coordinate transformations this equation can be written as

$$\frac{3\,m}{2}\;\ddot{x}+b\;\dot{x}+\frac{11\,k}{2}\;x=-\frac{M(t)}{\rho}.$$

b) For the numerical values given above (with consistent units), this equation reduces to

$$3\;\ddot{x}+2\;\dot{x}+88\;x=-8\;\sin(4\;t),$$

from which we can identify the appropriate parameters as:

$$\omega_n=\sqrt{\frac{88}{3}},\qquad \zeta=\frac{1}{\sqrt{264}},\qquad r=\sqrt{\frac{6}{11}}.$$

Therefore, the amplitude of the translational oscillations becomes

$$X=\frac{F}{k}\mathcal{M}(r,\zeta)=\frac{8}{88}\frac{1}{\sqrt{\left(1-\frac{6}{11}\right)^2+\left(2\frac{1}{\sqrt{264}}\sqrt{\frac{6}{11}}\right)^2}}=\frac{1}{\sqrt{26}}.$$

Likewise, the phase shift of the response is:

$$\tan\phi=\frac{2\,\zeta\,r}{1-r^2}=\frac{2\frac{1}{\sqrt{264}}\sqrt{\frac{6}{11}}}{1-\frac{6}{11}}=\frac{1}{5},$$

so that $\phi=0.20$ rad $=11.3°$.

c) In the development of the equation of motion, the friction force was eliminated by summing moments about C. Using linear momentum balance we can reintroduce the friction force as

$$\sum\boldsymbol{F}=\left(f_r-k\,x-2\,k\,z-b\,\dot{x}\right)\hat{\boldsymbol{\imath}}+\left(N-m\,g\right)=m\,\ddot{x}\;\hat{\boldsymbol{\imath}}=m\,{}^{\mathcal{F}}\mathbf{a}_G.$$

Therefore, solving for f_r we find that

$$f_r=m\,\ddot{x}+b\,\dot{x}+4\,k\,x.$$

With $x(t)$ represented as $x(t)=X\;\sin(\omega\,t-\phi)$, where X and ϕ are found above, the friction force becomes

$$f_r(t)=\left(4\,k-m\,\omega^2\right)X\;\sin(\omega\,t-\phi)+\left(b\;\omega\right)X\;\cos(\omega\,t-\phi).$$

The magnitude of the friction force is then found to be

$$\left\|f\right\| = X\sqrt{\left(4\,k - m\,\omega^2\right)^2 + \left(b\,\omega\right)^2}.$$

For these parameter values $\|f\| = 6.47$ N.

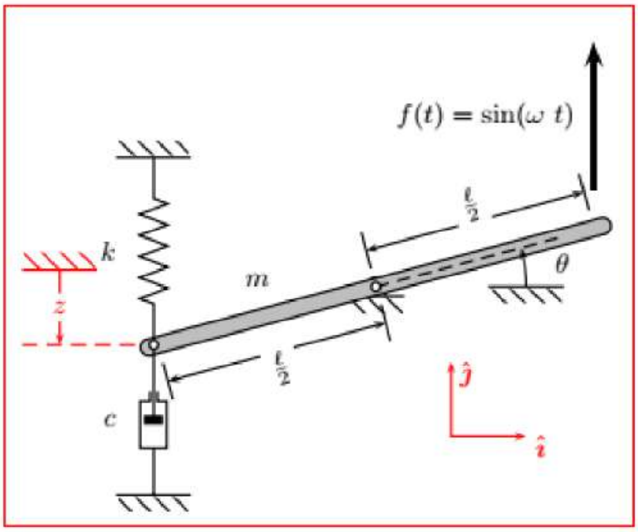
Problem: 02

For the mechanical system shown to the right, the uniform rigid bar has mass m and pinned at point O. For this system:

a) find the equations of motion (in terms of the given parameters—*do not substitute in numerical values yet*);

b) if $c = 0.25$ N/(m/s), $k = 32$ N/m, $m = 2$ kg, and $\ell = 0.25$ m, find the amplitude of the force transmitted to the ground through combination of the spring and damper when $\omega = 4$ rad/s.

c) if $c = 0$, $m = 2$ kg, and $\ell = 0.25$ m, find the value of the stiffness k so that the bar's amplitude of oscillation is less than $\pi/6$ rad for all forcing frequencies greater than 20 rad/s.

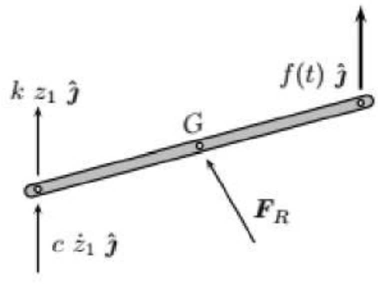


Solution

a) In addition to θ, we define the additional coordinate z, which measures the deflection at the left end of the bar, with θ and z related as:

$$z = \frac{\ell}{2}\,\theta$$

A free-body diagram for this system is shown to the right. Applying angular momentum balance on the bar eliminates the appearance of the reaction force and leads to:



$$\sum \boldsymbol{M}_G = I^G\,\ddot{\theta}\,\hat{\boldsymbol{k}},$$

$$\left(f(t) - k\,z - c\dot{z}\right)\frac{\ell}{2}\,\hat{\boldsymbol{k}} = \frac{m\,\ell^2}{12}\,\ddot{\theta}\,\hat{\boldsymbol{k}}.$$

Solving for z in terms of θ, the equation of motion becomes:

$$\frac{m\,\ell^2}{12}\,\ddot{\theta} + \frac{c\,\ell^2}{4}\,\dot{\theta} + \frac{k\,\ell^2}{4}\,\theta = \frac{\ell}{2}\,f(t).$$

b) In standard form, this equation of motion can be written as:

$$\ddot{\theta} + \left(\frac{3\,c}{m}\right)\,\dot{\theta} + \left(\frac{3\,k}{m}\right)\,\theta = \left(\frac{6}{m\,\ell}\right)\,\sin(\omega\,t),$$

so that:

$$\omega_n = \sqrt{\frac{3\,k}{m}}, \qquad \zeta = \frac{\sqrt{3}\,c}{2\,\sqrt{k\,m}}, \qquad M_0 = \frac{6}{m\,\ell}.$$

The amplitude of the moment transmitted to the ground can be written as:

$$\begin{aligned} M_T &= (m_{\text{eq}}\,M_0)\,\frac{\sqrt{1+(2\,\zeta\,r)^2}}{\sqrt{(1-r^2)^2+(2\,\zeta\,r)^2}}, \\ &= \frac{\ell}{2}\,\frac{\sqrt{1+\left(\frac{c\,\omega}{k}\right)^2}}{\sqrt{\left(1-\frac{m\,\omega^2}{3\,k}\right)^2+\left(\frac{c\,\omega}{k}\right)^2}} = \frac{\ell}{2}\,\frac{\sqrt{k^2+(c\,\omega)^2}}{\sqrt{\left(k-\frac{m\,\omega^2}{3}\right)^2+(c\,\omega)^2}}. \end{aligned}$$

The amplitude of the force transmitted to the ground is then $F_T = M_T/(\ell/2)$, or:

$$F_T = \frac{\sqrt{k^2+(c\,\omega)^2}}{\sqrt{\left(k-\frac{m\,\omega^2}{3}\right)^2+(c\,\omega)^2}}.$$

Substituting in the numerical values given in the problem statement, we find that:

$$F_T = 1.50\ \ \text{N}$$

c) The amplitude of the steady-state vibrations can be written as:

$$\begin{aligned} \Theta &= \frac{M_0}{\omega_n^2}\,\frac{1}{\sqrt{(1-r^2)^2+(2\,\zeta\,r)^2}}, \\ &= \frac{2}{k\,\ell}\,\frac{1}{\sqrt{\left(1-\frac{m\,\omega^2}{3\,k}\right)^2+\left(\frac{c\,\omega}{k}\right)^2}} = \frac{2}{\ell}\,\frac{1}{\sqrt{\left(k-\frac{m\,\omega^2}{3}\right)^2+(c\,\omega)^2}}. \end{aligned}$$

Substituting in the numerical values given in the problem statement, we find that:

$$\Theta = \frac{8}{\left|\left(k-\frac{2\,\omega^2}{3}\right)\right|}$$

Therefore, if the amplitude of vibration is less than $\pi/6$:

$$\begin{aligned} \frac{8}{\left|k-\frac{2\,\omega^2}{3}\right|} &\le \frac{\pi}{6}, \\ \frac{48}{\pi} &\le \left|k-\frac{2\,\omega^2}{3}\right|. \end{aligned}$$

This inequality has two solutions:

$$k \geq \frac{48}{\pi} + \frac{2\,\omega^2}{3}, \qquad k \leq \frac{2\,\omega^2}{3} - \frac{48}{\pi}.$$

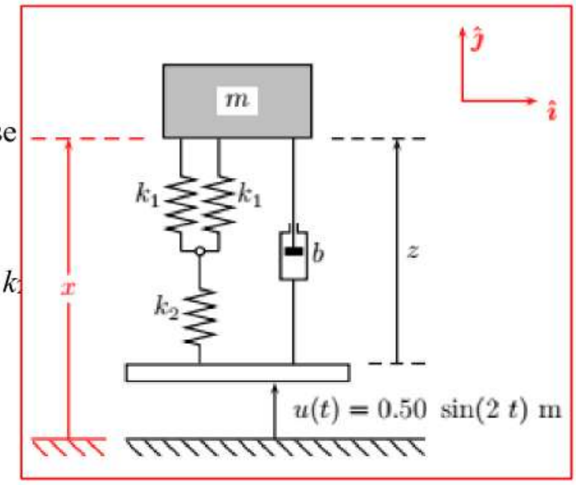
Since this condition must be satisfied for all $\omega \geq 20$ rad/s, we take the second inequality and find that:

$$k \leq \frac{2\,(20)^2}{3} - \frac{48}{\pi} = 251.$$

Problem: 03

The system shown to the right is subject to base Find the steady-state response of the system in terms of z, with:

$m = 2.0$ kg, $b = 4.0$ N/(m/s), $k_1 = 3.00$ N/m, k N/m.



Solution

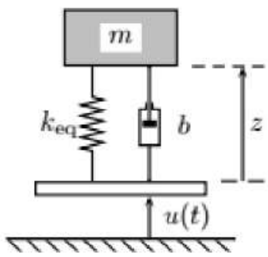
a) We define the addition coordinate x which measures the absolute displacement of the mass with respect to the ground, so that:

$$x = z + u(t).$$

Notice that the collection of springs can be replaced by a single equivalent spring, with:

$$k_{eq} = \frac{1}{\frac{1}{2\,k_1} + \frac{1}{k_2}} = \frac{2\,k_1\,k_2}{2\,k_1 + k_2} = 4 \text{ N/m}.$$

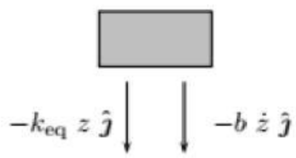
The new equivalent system is shown to the right.



An appropriate free-body diagram is shown to the right. In terms of the identified coordinates, the acceleration of the mass center is:

$${}^{\mathcal{F}}\mathbf{a}_G = \ddot{x}\,\hat{\jmath} = (\ddot{z} + \ddot{u})\,\hat{\jmath},$$

with $\ddot{u}(t) = -(u_0\,\omega^2)\,\sin(\omega\,t)$.



Therefore, linear momentum balance on the mass yields:

$$\begin{aligned} \sum \boldsymbol{F} &= m\,{}^{\mathcal{F}}\mathbf{a}_G, \\ \Big(-k_{eq}\,z - b\,\dot{z}\Big)\,\hat{\jmath} &= m\,\ddot{x}\,\hat{\jmath} \end{aligned}$$

with:

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m}}, \qquad \zeta = \frac{b}{2\sqrt{k_{\text{eq}}\ m}}.$$

Therefore the steady state response of this system becomes:

$$z(t) = Z\ \sin(\omega\ t - \psi),$$

with $Z = u_0\ \Lambda(r, \zeta)$, and:

$$\Lambda = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\ \zeta\ r)^2}}, \qquad \tan\psi = \frac{2\ \zeta\ r}{1-r^2}, \qquad \text{and } r = \frac{\omega}{\omega_n}$$

For the numerical values of this problem:

$$\omega_n = \sqrt{2}, \qquad \zeta = \frac{1}{\sqrt{2}}, \qquad r = \sqrt{2},$$

so that:

$$\Lambda = \frac{2}{\sqrt{5}}, \qquad \tan\psi = \frac{2}{-1}$$

Recall that the phase shift ψ must be positive, so that is $\tan\psi$ is negative, then ψ is in the second quadrant, so that $\psi = 3.03$ rad. Finally:

$$z(t) = \frac{1}{\sqrt{5}}\ \sin(2\ t - 2.03).$$

Problem: 04

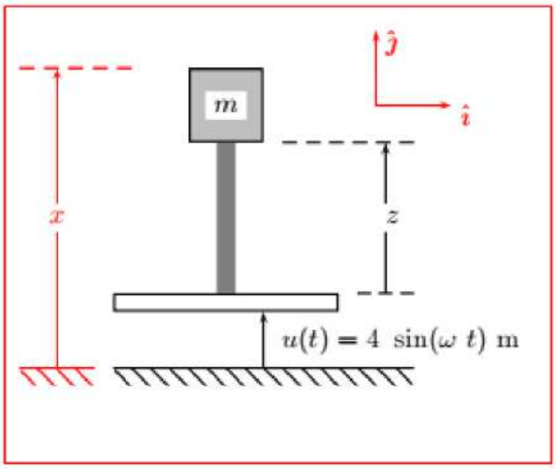
The mass $m = 2$ kg is supported by an elastic cantilever beam attached to a foundation which undergoes harmonic motion of the form:

$$u(t) = 4\sin(\omega t)\ \text{m},$$

If the beam has length $l = 20$ cm, while $AE = 16$ N:

a) find the equations of motion in terms of z, the relative displacement between the mass and the foundation (assume the beam has zero mass);

b) what is the amplitude of the resulting motion in terms of the forcing frequency ω?



Solution

a) The equivalent spring for this cantilever beam is:

$$k_{\text{eq}} = \frac{AE}{l} = \frac{16\text{ N}}{0.2\text{ m}} = 80\text{ N/m}.$$

The acceleration of the block with respect to the ground is ${}^{\mathcal{F}}\mathbf{a}_G = (\ddot{u} + \ddot{z})\hat{\jmath}$, so that, in terms of z, the equations of motion become:

$$\begin{aligned} m\ddot{z} + k_{\text{eq}}\, z &= -m\ddot{u}, \\ \ddot{z} + 40z &= 4\omega^2 \sin(\omega t). \end{aligned}$$

b) For this undamped system, the amplitude of the resulting steady-state motion is:

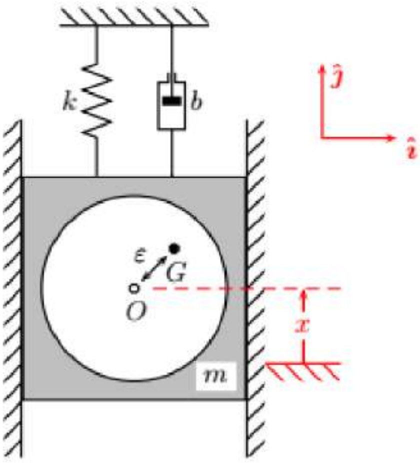
$$\begin{aligned} X &= \frac{4\omega^2}{40} \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{40}\right)^2}}, \\ &= \frac{4\omega^2}{|40 - \omega^2|}. \end{aligned}$$

Problem: 05

The unbalanced rotor shown in the figure is pinned to a frame and supported by a spring and damper. If the total mass is m while the mass center G is located at an eccentricity of ε from the the center of rotation O,

a) find the damped natural frequency;

b) what is the steady-state amplitude of vibration when the rotor spins at this angular speed.



Solution

We define $x(t)$ as the vertical displacement of the geometric center of the rotor as measured from static equilibrium. As a result, the mass center G is described by the position $z(t) = x(t) + \varepsilon \sin(\omega t)$. Note that ε measures the eccentricity of the *mass center*, not the location of the mass imbalance. Consequently, the governing equations of motion can be written:

$$\begin{aligned} m\ddot{z} &= -kx - b\dot{x}, \\ m\ddot{x} - \varepsilon\omega^2 \sin(\omega t) &= -kx - b\dot{x}, \end{aligned}$$

or in more standard form:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \varepsilon\omega^2 \sin(\omega t).$$

a) In the above system we find $\omega_n = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{b}{2\sqrt{km}}$, so that the damped natural frequency can be written:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$

b) For an arbitrary forcing frequency the amplitude of oscillation is $A = \varepsilon\Lambda$, where:

$$\Lambda = \frac{\omega^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}},$$

which, with $\omega = \omega_d$, reduces the amplitude to:

$$A = \varepsilon\frac{1-\zeta^2}{\zeta\sqrt{4-3\zeta^2}},$$

with ζ defined above.

Chapter 5:

Free oscillations of systems with several degrees of freedom

1. *Introduction:*

Multi-degree-of-freedom systems consist of several coupled single-degree-of-freedom systems. The number of degrees of freedom determines the number of differential equations governing the evolution over time of these coordinates.

In systems with two degrees of freedom, there are two coordinates that characterize the vibrational motion.

There are three types of coupling: elastic, inertial and viscous.

2. *Differential equations:*

For the study of systems with two undamped and unforced degrees of freedom, it is necessary to write two differential equations of motion that can be obtained from the Lagrange equations:

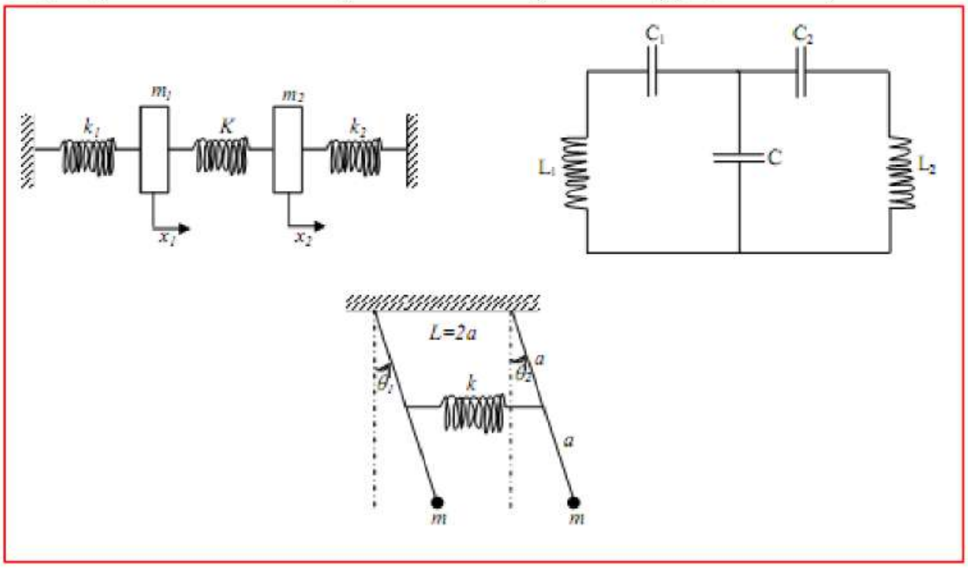
$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \left(\frac{\partial L}{\partial q_1}\right) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \left(\frac{\partial L}{\partial q_2}\right) = 0 \end{cases}$$

q_1 and q_2 are the two generalized coordinates which characterize the system with two degrees of freedom.

3. *Coupling types:*

3.1. Coupling by elasticity:

The coupling between the two systems is through a spring (capacitance).

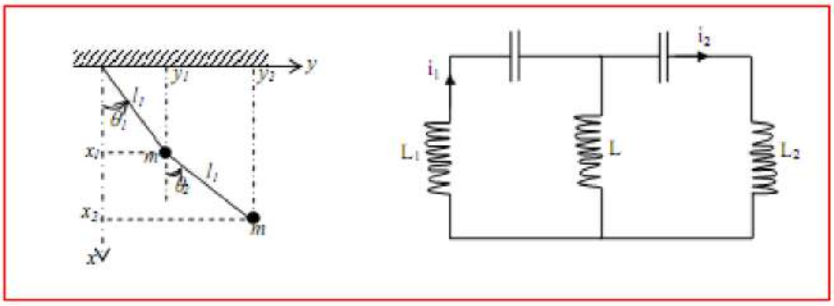


The corresponding differential equations are:

$$\begin{cases} \ddot{x}_1 + 2\,\lambda_1\dot{x}_1 + w^2x_1 = a_1\,x_2 \\ \ddot{x}_2 + 2\,\lambda_2\dot{x}_2 + w^2x_2 = a_2\,x_1 \end{cases}$$ Such as : $a_1\,x_2$ And $a_2\,x_1$ are the coupling terms a_1 and a_2 are constants .

3.2. Inertial coupling:

The coupling between the two systems is through a mass (coil).

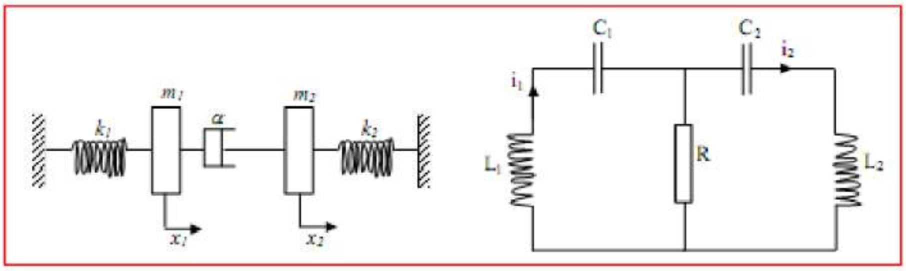


The corresponding differential equations are:

$$\begin{cases} \ddot{x}_1 + 2\,\lambda_1\dot{x}_1 + w^2x_1 = c_1\,\ddot{x}_2 \\ \ddot{x}_2 + 2\,\lambda_2\dot{x}_2 + w^2x_2 = c_2\,\ddot{x}_1 \end{cases}$$ Such as : $c_1\,\ddot{x}_2$ and $c_2\,\ddot{x}_1$ are the coupling terms c_1 and c_2 are constants.

3.3. Viscous coupling:

The coupling between the two systems is through a damper (resistor).



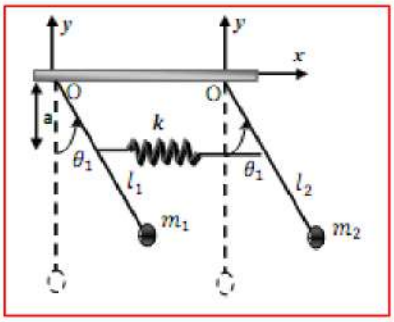
The corresponding differential equations are:

$\begin{cases} \ddot{x}_1 + 2\,\lambda_1 \dot{x}_1 + \ w^2 x_1 \ = b_1\,\dot{x}_2 \\ \ddot{x}_2 + 2\,\lambda_2 \dot{x}_2 + \ w^2 x_2 \ = b_2\,\dot{x}_1 \end{cases}$ Such as : $b_1\,\dot{x}_2$ and $b_2\,\dot{x}_1$ are the coupling terms b_1 and b_2 are constants.

4. *Examples of 2 DOF systems :*

Example: Coupled pendulums: (Elastic Coupling):

Consider two pendulums which are coupled by a horizontal spring of stiffness constant k at a distance a from the axis of rotation.



4.1 Differential equations of motion:

- **The coordinates of the system elements:**

The mass m_1 is at a distance l_1 of O.

$$m_1 \begin{cases} x_{m_1} = l_1 sin\theta_1 \\ y_{m_1} = -l_1 cos\theta_1 \end{cases} \Rightarrow m_1 \begin{cases} \dot{x_{m_1}} = l_1\dot{\theta}_1 cos\theta_1 \\ \dot{y_{m_1}} = l_1\dot{\theta}_1 sin\theta_1 \end{cases} \Rightarrow= v_{m1}^2 = l_1^2\dot{\theta}_1^2$$

The mass m_2 is at a distance l_2 of O.

$$m_2 \begin{cases} x_{m_2} = l_2 sin\theta_2 \\ y_{m_2} = -l_2 cos\theta_2 \end{cases} \Rightarrow m_1 \begin{cases} \dot{x_{m_2}} = l_2\dot{\theta}_2 cos\theta_2 \\ \dot{y_{m_2}} = l_2\dot{\theta}_2 sin\theta_2 \end{cases} \Rightarrow= v_{m2}^2 = l_2^2\dot{\theta}_2^2$$

$$k = (a\,sin\theta_1 - a\,sin\theta_2) = a(\,sin\theta_1 - \,sin\theta_2)$$

- **The kinetic energy of the system:**

$$T_m = T_{m_1} + T_{m_2} = \frac{1}{2} m_1 v_{m1}^2 + \frac{1}{2} m_2 v_{m2}^2 \ldots (1)$$

- $T_{m_1} = \frac{1}{2} m_1 v_{m1}^2 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$
- $T_{m_2} = \frac{1}{2} m_2 v_{m2}^2 = \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2$

So equation (1) becomes : $T_m = T_{m_1} + T_{m_2} = \frac{1}{2}(m_1 l_1^2 \dot{\theta}_1^2 + m_2 l_2^2 \dot{\theta}_2^2)$.

- **The potential energy of the system:**

$$U = U_K + U_{m1} + U_{m2} \ldots (2)$$

If we choose the axis as the origin of the potential energies (Ox) we have for the two masses:

$U_{m1}+U_{m2}=-m_1gl_1cos\theta_1-m_2gl_2cos\theta_2$ (The minus sign comes from the fact that the mass m is below the chosen axis). And $U_K=\frac{1}{2}ka^2(\ sin\theta_1-\ sin\theta_2)^2$

So equation (2) becomes:

$$U=\frac{1}{2}ka^2(\ sin\theta_1-\ sin\theta_2)^2-m_1gl_1cos\theta_1-m_2gl_2cos\theta_2$$

- The Lagrange function will therefore be:

$$L=T-V=\frac{1}{2}(m_1l_1^2\dot{\theta}_1^2+m_2l_2^2\dot{\theta}_2^2)-\frac{1}{2}Ka^2(\ sin\theta_1-\ sin\theta_2)^2+m_1gl_1cos\theta_1+m_2gl_2cos\theta_2$$

Two Lagrange equations are necessary to describe the motion:

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_1}}\right)-\left(\frac{\partial L}{\partial\theta_1}\right)=0\ ...(3)\\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_2}}\right)-\left(\frac{\partial L}{\partial\theta_2}\right)=0\ ...(4)\end{cases}$$

$$(3)\text{ devient}\begin{cases}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_1}}\right)=m_1l_1^2\ddot{\theta_1}\\ \left(\frac{\partial L}{\partial\theta_1}\right)=-Ka^2\ cos\theta_1(\ sin\theta_1-\ sin\theta_2)\ -m_1gl_1sin\theta_1\end{cases}\Rightarrow m_1l_1^2\ddot{\theta_1}$$
$$+Ka^2\ cos\theta_1(\ sin\theta_1-\ sin\theta_2)+m_1gl_1sin\theta_1=0.(5)$$

In the case of weak oscillations, the angles are very small, we have:

$$\begin{cases}sin(\theta)\approx\theta\\ cos(\theta)\approx 1-\frac{\theta^2}{2}\ \approx 1\end{cases}$$

So the equation (5) : $m_1l_1^2\ddot{\theta_1}+Ka^2(\ \theta_1-\theta_2)+m_1gl_1\theta_1=0$

So the 02 differential equations of motion are:

$$\begin{cases}m_1l_1^2\ddot{\theta_1}+(Ka^2+m_1gl_1)\theta_1=Ka^2\theta_2\ ...(6)\\ m_2l_2^2\ddot{\theta_2}+(Ka^2+m_2gl_2)\theta_2=Ka^2\theta_1\ ...(7)\end{cases}$$

Noticed :

- The coupling term ka^2 is a function of k therefore the coupling is elastic.
- If $a=0$ Or $K=0\Rightarrow$ zero coupling: the two systems are independent **.**
- The two differential equations have **02** solutions: $\theta_1(t)$ And $\theta_2(t)$.

We assume that the system admits harmonic solutions:

$\theta_1(t)=A_1\ cos(wt+\varphi)$ And $\theta_2(t)=A_2\ cos(wt+\varphi')$

Such as: A_1, A_2, $\varphi$and φ', w is one of the system's own pulsations.

$$\begin{cases} \theta_1(t) = A_1 \cos(wt + \varphi) \Rightarrow \ddot{\theta}_1 = -w^2\,\theta_1 \\ \theta_2(t) = A_2 \cos(wt + \varphi') \Rightarrow \ddot{\theta}_2 = -w^2\,\theta_2 \end{cases}$$

We replace in equations (6) and (7) we obtain **:**

$$\begin{cases} (Ka^2 + m_1 g l_1 - m_1 l_1^2 w^2)\theta_1 - Ka^2\theta_2 = 0 \dots (8) \\ (Ka^2 + m_2 g l_2 - m_2 l_2^2 w^2)\theta_2 - Ka^2\theta_1 = 0 \dots (9) \end{cases}$$

4.2 Calculation of own pulsations:

We suppose that $m_2 = m_2 = m, l_2 = l_2 = l$.

$$\begin{cases} (8) \\ (9) \end{cases} \Rightarrow \begin{pmatrix} Ka^2 + mgl - ml^2w^2 & -Ka^2 \\ -Ka^2 & Ka^2 + mgl - ml^2w^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

These two equations will accept a solution if the determinant$= 0$

$$\begin{vmatrix} Ka^2 + mgl - ml^2w^2 & -Ka^2 \\ -Ka^2 & Ka^2 + mgl - ml^2w^2 \end{vmatrix} = 0$$

$$\Rightarrow (Ka^2 + mgl - ml^2w^2)^2 - (Ka^2)^2 = 0 \Rightarrow Ka^2 + mgl - ml^2w^2 = \begin{cases} +Ka^2 \\ -Ka^2 \end{cases}$$

$$\Rightarrow \begin{cases} w_1^2 = \frac{g}{l} + 2\left(\frac{k}{m}\right)\left(\frac{a}{l}\right)^2 \\ w_2^2 = \frac{g}{l} \end{cases} \text{tels que:} \begin{cases} w_1\text{: la 1er pulsation propre.} \\ w_2\text{: la 2eme pulsation propre.} \end{cases}$$

Noticed :

- If $a = 0$ Or $k = 0$, the coupling is zero$\Rightarrow w_1^2 = w_2^2 = \frac{g}{l}$.
- When the system oscillates with one of its 02 pulsations we say that the system oscillates in one of its two modes.

4.3 Oscillation modes:

The mode is the state in which the dynamic elements of the system perform a harmonic oscillation with the same pulsation which corresponds to one of its two pulsations.

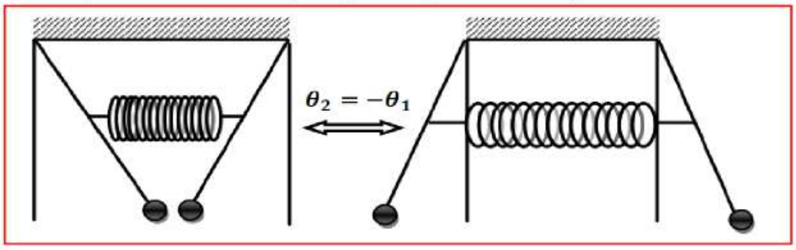
Calculation of oscillation modes:

In each mode the two masses perform simple harmonic movements with the same pulsation (w_1 ou w_2)and the two pendulums pass through the equilibrium position at the same instant.

__First mode__ : We replace $w_1^2 = \frac{g}{l} + 2\left(\frac{k}{m}\right)\left(\frac{a}{l}\right)^2$ in the equation **(8)** Or **(9)** ; we obtain :$\theta_2 = -\theta_1$

Noticed :

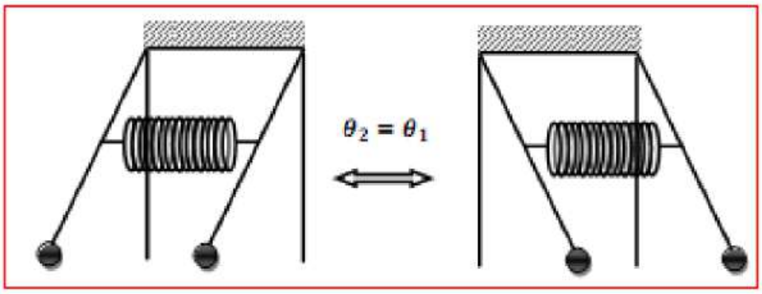
- In the first mode the two pendulums have the same pulsation w_1, the same amplitude and a phase shift π.
- The two pendulums have opposite movements.
- Elongation and compression of the spring each period except in the middle of the spring.



__Second mode__ : We replace $w_2^2 = \frac{g}{l}$ in the equation **(8)** Or **(9)** ; we obtain :$\theta_2 = \theta_1$

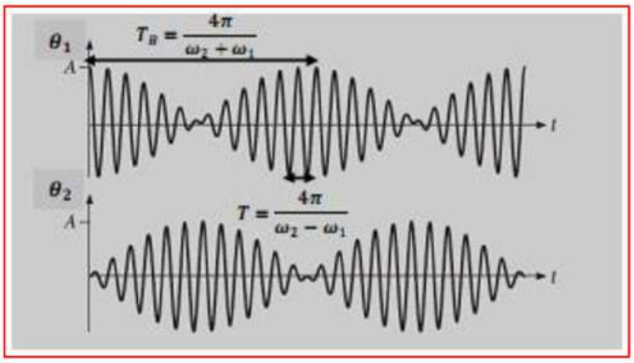
Noticed :

- T he two pendulums move in the same direction (oscillation in phase).
- The spring does not undergo any variation in its length.



4.4 Beating phenomenon:

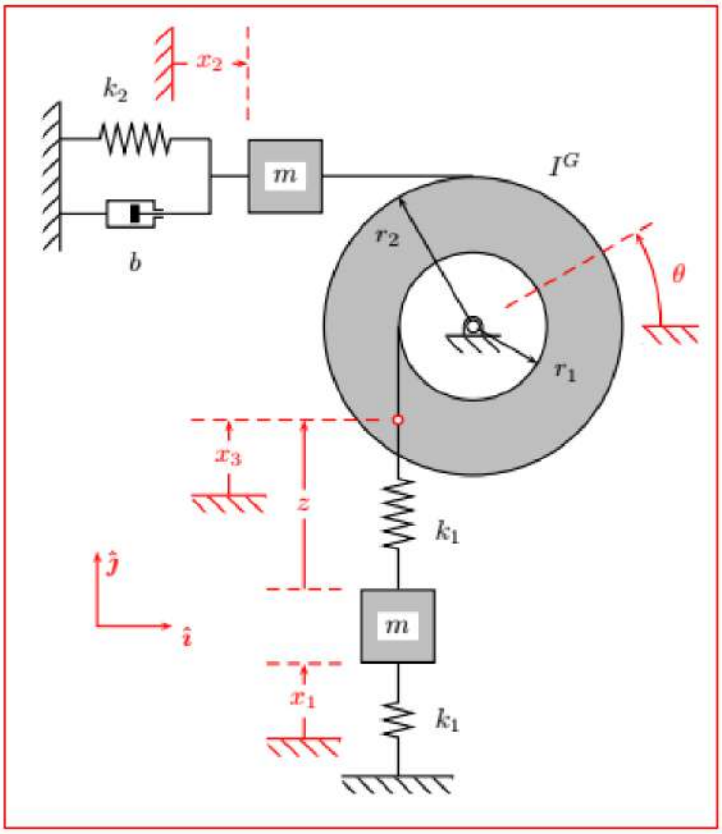
When coupling is weak(k) weak , the own pulsations of the 2 oscillators$(w_1$ et $w_2)$ are close , $(w_1 \approx w_2) \Rightarrow \Delta w = w_1 - w_2$ is low, a beating phenomenon occurs. The 2 oscillators transmit energy between them and vibrate with a pulsation$w$ equal to the average of the two proper pulsations$w = \frac{1}{2}(w_1 + w_2)$, with a period equal to $T = \frac{2\pi}{w} = \frac{4\pi}{w_1 + w_2}$ While the pulsation of the beat is equal to: $w_B = \frac{1}{2}(w_2 - w_1),$ with a period $T_B = \frac{4\pi}{w_2 - w_1}$.



Problem corrected

Problem: 01

In the figure shown to the right, in the ab- sence of gravity the springs are unstretched in the equilibrium position. Determine the equations of motion for this system.

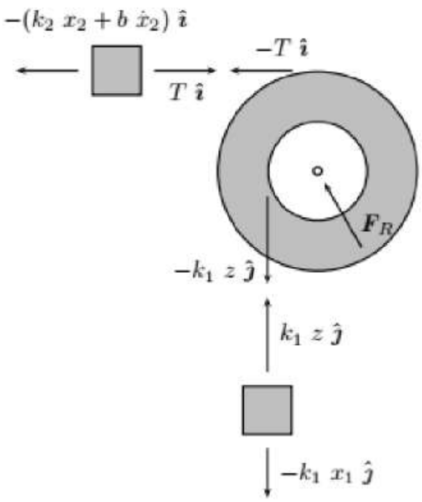


Solution:

Because we can, we define five different coordinates to describe the dynamical behavior of this two degree-of-freedom system, leading to the following transformations:

$$x_2 = -r_2\ \theta, \qquad x_3 = -r_1\ \theta, \qquad z = x_3 - x_1.$$

A free-body diagram for this system is shown to the right. We develop three equations of motion based on linear momentum balance on both blocks and angular momentum balance on the disk:



Block 1:

$$\begin{aligned} \sum \boldsymbol{F} &= m\ ^{\mathcal{F}}\mathbf{a}_{G_1}, \\ \Big(k_1\ z - k_1\ x_1\Big)\ \hat{\jmath} &= m\ \ddot{x}_1\ \hat{\jmath} \end{aligned}$$

Block 2:

$$\begin{aligned} \sum \boldsymbol{F} &= m\ ^{\mathcal{F}}\mathbf{a}_{G_2}, \\ \Big(T - b\ \dot{x}_2 - k_2\ x_2\Big)\ \hat{\imath} &= m\ \ddot{x}_2\ \hat{\imath} \end{aligned}$$

Disk:

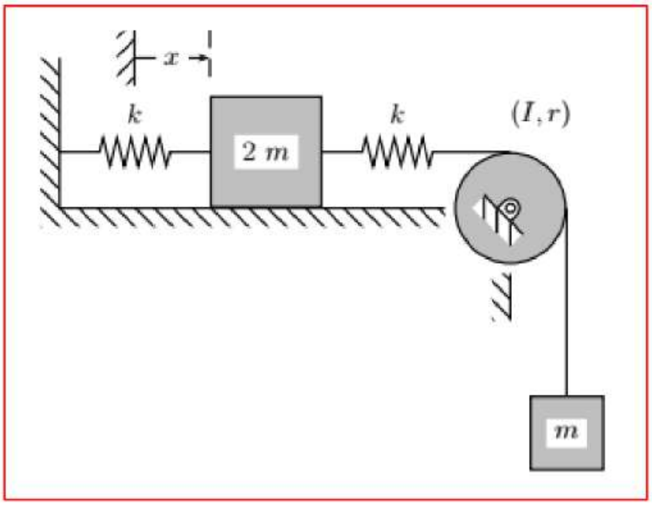
$$\begin{aligned} \sum \boldsymbol{M}_G &= I^G\ \boldsymbol{\alpha}_{D/\mathcal{F}}, \\ \Big(T\ r_2 + k_1\ r_1\ z\Big)\ \hat{\boldsymbol{k}} &= I^G\ \ddot{\theta}\ \hat{\boldsymbol{k}} \end{aligned}$$

Problem: 02

For the system shown in the figure:

a) what is the degree-of-freedom for this system?

b) using Lagrange's equations, determine the differential equations that govern the motion.



Solution:

a) This system contains three masses which are each allowed to move in only one direc- tion. The upper mass slides horizontally with displacement x_1, while the disk rotates through an angle θ. Finally, the suspended mass moves vertically and its position can be described by the coordinate x_2. However, because the disk and the suspended mass are connected by an inextensible string, their motion can be related by:

$$x_2 = r\theta.$$

So this system has only two independent coordinates and therefore it is a two-degree- of-freedom system.

b) We utilize the coordinates x_1, x_2, and θ, as measured from the unstretched position of the two springs. Therefore, the kinetic and potential energies are written as:

$$\mathcal{T} = \frac{1}{2}(2m)\,\dot{x}_1^2 + \frac{1}{2}\,m\,\dot{x}_2^2 + \frac{1}{2}\,I\,\dot{\theta}^2,$$
$$\mathcal{V} = \frac{1}{2}\,k\,x_1^2 + \frac{1}{2}\,k\,(r\theta - x_1)^2 - m\,g\,x_2.$$

However, x_2 and θ are related by the above relationship. Thus eliminating θ, the energies become:

$$\mathcal{T} = \frac{1}{2}\,(2\,m)\,\dot{x}_1^2 + \frac{1}{2}\,m\,\dot{x}_2^2 + \frac{1}{2}\,\frac{I}{r^2}\,\dot{x}_2^2,$$
$$\mathcal{V} = \frac{1}{2}\,k\,x_1^2 + \frac{1}{2}\,k\,(x_2 - x_1)^2 - m\,g\,x_2,$$

which, using Lagrange's equations, yields the equations of motion:

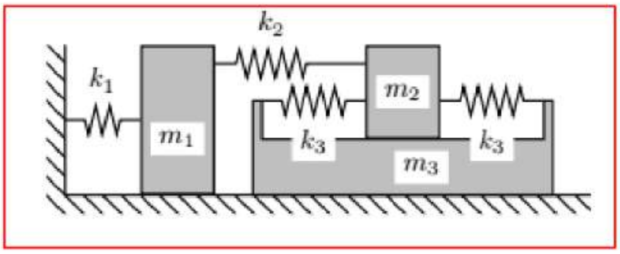
$$2m\,\ddot{x}_1 + 2\,k\,x_1 - k\,x_2 = 0,$$
$$\left(m + \frac{I}{r^2}\right)\ddot{x}_2 - k\,x_1 + k\,x_2 = m\,g.$$

Problem: 03

For the system shown in the figure:

a) find the mass and the stiffness matrix;

b) is your system of equations dynamically coupled, statically coupled, or both?



Solution

We choose the coordinates (x_1, x_2, x_3), which represent the positions of the three masses.

a) To determine the stiffness matrix, we use the stiffness influence coefficients. Maintaining a unit displacement of each mass in turn requires forces of the form:

$$(x_1, x_2, x_3) = (1,0,0) \rightarrow \boldsymbol{f} = (k_1 + k_2, -k_2, 0)^T,$$
$$(x_1, x_2, x_3) = (0,1,0) \rightarrow \boldsymbol{f} = (-k_2, k_2 + 2k_3, -2k_3)^T,$$
$$(x_1, x_2, x_3) = (0,0,1) \rightarrow \boldsymbol{f} = (0, -2k_3, 2k_3)^T.$$

Therefore, with these coordinates the stiffness matrix is:

$$\boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + 2k_3 & -2k_3 \\ 0 & -2k_3 & 2k_3 \end{bmatrix}$$

Alternatively, we can define the potential energy of the system as:

$$\begin{aligned} \mathcal{V} &= \frac{1}{2}(k_1)(x_1)^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}(2k_3)(x_3 - x_2)^2, \\ &= \frac{1}{2}(k_1 + k_2)(x_1)^2 - \frac{1}{2}(2k_2)(x_1 x_2) + \frac{1}{2}(k_2 + 2k_3)(x_2)^2 \\ &\quad -\frac{1}{2}(2k_3)(x_2 x_3) + \frac{1}{2}(2k_3)(x_3)^2, \end{aligned}$$

which leads to the same stiffness matrix.

To determine the mass matrix, we could use the inertia influence coefficients, but, for variety, we determine the kinetic energy as:

$$\mathcal{T} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2.$$

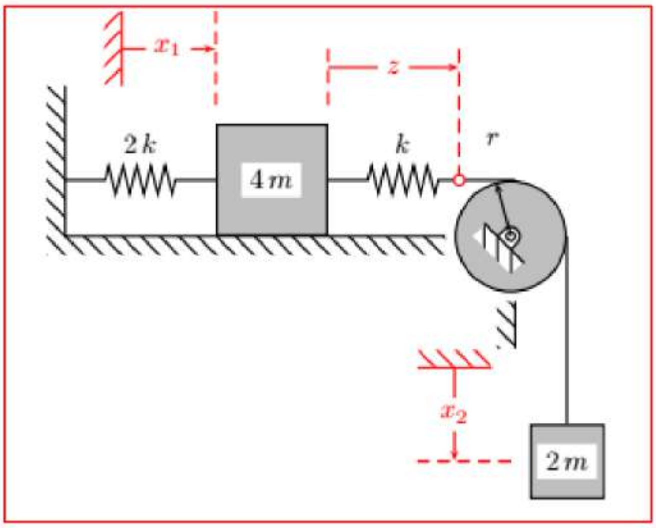
Therefore, the mass matrix is:

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

b) With this choice of coordinates, the mass matrix is diagonal and the stiffness matrix contains nonzero off-diagonal terms. Thus, the system is statically coupled but dynamically uncoupled.

Problem: 04

In the multi-degree-of-freedom system shown in the figure, the block with mass $4m$ slides on a smooth, frictionless surface. If the pulley is massless:



a) using Lagrange's equations, determine the differential equations governing the motion, as measured from static equilibrium;

a) with m = 1 kg and k = 16 N/m, find the natural frequencies and mode shapes for the free vibration of this sys- tem. Normalize the mode shapes so that with respect to the mass matrix the amplitude of each mode is one;

find the general solution to these equa- tions for the above values of m and k.

Solution

---

a) We identify the three coordiantes x_1, x_2, and z, with

$$z = x_2 - x_1.$$

Measuring the response from static equilibria and neglecting the gravitational potential energy, the kinetic and potential energies for this system can be written as

$$\begin{aligned} \mathcal{T} &= \frac{1}{2}(4\,m)\,\dot{x}_1^2 + \frac{1}{2}(2\,m)\,\dot{x}_2^2, \\ \mathcal{V} &= \frac{1}{2}(2\,k)\,x_1^2 + \frac{1}{2}(k)\,z^2, \end{aligned}$$

In terms of x_1 and x_2, the potential energy becomes

$$\mathcal{V} = \frac{1}{2}(2\,k)\,x_1^2 + \frac{1}{2}(k)\,(x_2 - x_1)^2 = \frac{1}{2}(3\,k)\,x_1^2 + \frac{1}{2}(-2\,k)\,x_1\,x_2 + \frac{1}{2}(k)\,x_2^2.$$

Therefore the equations of motion become

$$\begin{aligned} 4\,m\,\ddot{x}_1 + 3\,k\,x_1 - k\,x_2 &= 0, \\ 2\,m\,\ddot{x}_2 - k\,x_1 + k\,x_2 &= 0, \end{aligned}$$

or in matrix form

$$m\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + k\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{0}.$$

b) The corresponding eigenvalue problem for the above system is

$$(\boldsymbol{M}^{-1}\,\boldsymbol{K})\;\boldsymbol{u} = \lambda\,\boldsymbol{u} \qquad \longrightarrow \qquad \frac{k}{m}\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{k}{m}\,(\beta\,\boldsymbol{u})$$

The characteristic equation is

$$\left(\frac{3}{4}-\beta\right)\left(\frac{1}{2}-\beta\right)-\frac{1}{8} = \beta^2 - \frac{5}{4}\,\beta + \frac{1}{4} = 0,$$

with the solution

$$\beta = \frac{5 \pm 3}{8} \qquad \longrightarrow \qquad \lambda = \omega^2 = \left\{\frac{k}{4\,m},\, \frac{k}{m}\right\}.$$

Returning this to the eigenvalue problem, the mode shapes are defined by the equation

$$\frac{3}{4}\,u_{i1} + \frac{1}{4}\,u_{i2} = \beta_i\,u_{i1},$$

so that

$$\boldsymbol{u}_i = \begin{bmatrix} 1 \\ 4\,\beta_i - 3 \end{bmatrix} c_i \qquad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} c_1, \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} c_2.$$

Normalizing $\boldsymbol{u}_i$ by the mass matrix implies that

$$1 = \boldsymbol{u}_i^T\,\boldsymbol{M}\,\boldsymbol{u}_i = c_i^2\,\begin{bmatrix} 1 & (4\,\beta_i - 3) \end{bmatrix}\begin{bmatrix} 4\,m & 0 \\ 0 & 2\,m \end{bmatrix}\begin{bmatrix} 1 \\ (4\,\beta_i - 3) \end{bmatrix}.$$

Solving for c_i

$$c_i = \sqrt{\frac{1}{2\,m\,(2 + (4\,\beta_i - 3)^2)}} \qquad \longrightarrow \qquad c_1 = \sqrt{\frac{1}{12\,m}},\; c_2 = \sqrt{\frac{1}{6\,m}}$$

Finally, the normalized eigenpairs are

$$\omega_1 = \sqrt{\frac{k}{4\,m}}, \quad \boldsymbol{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{12\,m}} \\ -\frac{2}{\sqrt{12\,m}} \end{bmatrix}, \qquad \omega_2 = \sqrt{\frac{k}{m}}, \quad \boldsymbol{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{6\,m}} \\ \frac{1}{\sqrt{6\,m}} \end{bmatrix}.$$

c) With the above mode shapes and natural frequencies the general solution becomes

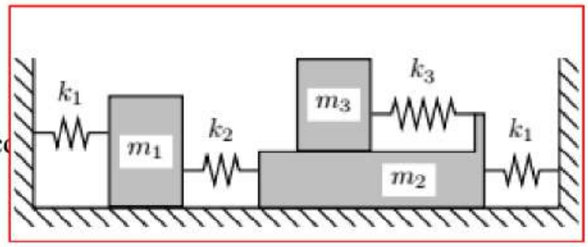
$$
\begin{aligned}
\boldsymbol{q}(t) = \left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right] &= \sum_{i=1}^{2} \left(A_i \sin(\omega_i t) + B_i \cos(\omega_i t)\right) \boldsymbol{u}_i, \\
&= \left(A_1 \sin\left(\sqrt{\frac{k}{4m}}\, t\right) + B_1 \cos\left(\sqrt{\frac{k}{4m}}\, t\right)\right) \left[\begin{array}{c} \frac{1}{\sqrt{12\, m}} \\ -\frac{2}{\sqrt{12\, m}} \end{array}\right] \\
&\quad + \left(A_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right) + B_2 \cos\left(\sqrt{\frac{k}{m}}\, t\right)\right) \left[\begin{array}{c} \frac{1}{\sqrt{6 m}} \\ \frac{1}{\sqrt{6 m}} \end{array}\right].
\end{aligned}
$$

Problem: 05

For the system shown in the figure:

a) find the mass and the stiffness matrices;

b) is your system of equations dynamically coupled, statically coupled, or both?



Solution

a) For coordinates we choose (x_1, x_2, x_3) as the displacements of each mass with respect to inertial space. Using influence coefficients, we find that:

$$
\boldsymbol{M} = \left[\begin{array}{ccc} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{array}\right], \qquad \boldsymbol{K} = \left[\begin{array}{ccc} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_1 + 2k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{array}\right]
$$

Alternatively, if we choose coordinates (x_1, x_2, z), where z represents the stretch in the spring connecting m_2 and m_3, we can determine the mass and stiffness matrices from the Lagrangian. The kinetic and potential energies are:

$$
\begin{aligned}
\mathcal{T} &= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 (\dot{x}_2 - \dot{x}_3)^2, \\
\mathcal{V} &= \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_2 z^2 + \frac{1}{2} k_1 x_2^2.
\end{aligned}
$$

Thus, with these coordinates the mass and stiffness matrices become:

$$
\boldsymbol{M} = \left[\begin{array}{ccc} m_1 & 0 & 0 \\ 0 & m_2 + m_3 & -m_3 \\ 0 & -m_3 & m_3 \end{array}\right], \qquad \boldsymbol{K} = \left[\begin{array}{ccc} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_1 + k_2 & 0 \\ 0 & 0 & k_2 \end{array}\right]
$$

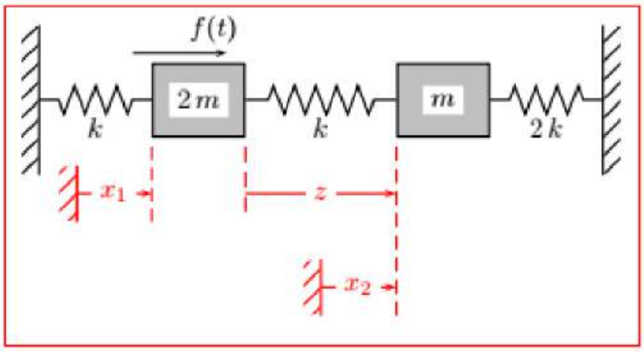
a) With the first choice of coordinates, the mass matrix is diagonal while the stiffness matrix is not, the system is statically coupled but dynamically uncoupled. With the latter coordinates neither matrix is diagonal so that the system is both statically and dynamically coupled.

Problem: 06

The two-degree-of-freedom system shown is subject to a harmonic force applied to the block of mass $2m$, of the form:

$$f(t) = (2\sin(t) \ \ \mathrm{N})\ \hat{\imath}.$$

If the mass and stiffness of the system are assumed to be m = 2 kg, and k = 4 N/m, find:



a) the equations of motion;

b) the forced, damped equation (single- degree-of-freedom) that describes the motion of each mode;

Solution

a) We define the coordinates x_1, x_2, and z, which are related as

$$z = x_2 - x_1.$$

The equations of motion become

$$m\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + k\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} f(t) \\ 0 \end{bmatrix}.$$

b) The corresponding eigenvalue problem can be written as

$$\left(\boldsymbol{M}^{-1}\boldsymbol{K}\right) = \frac{k}{m}\begin{bmatrix} 1 & -\frac{1}{2} \\ -1 & 3 \end{bmatrix}\boldsymbol{u} = \left(\frac{k}{m}\beta\right)\boldsymbol{u}$$

and the characteristic equation becomes

$$(1-\beta)(3-\beta) - \frac{1}{2} = \beta^2 - 4\beta + \frac{5}{2} = 0.$$

This quadratic equation has the solutions

$$\beta = 2 \pm \sqrt{\frac{3}{2}}, \qquad \longrightarrow \qquad \omega^2 = \left\{\frac{(\sqrt{8}-\sqrt{3})\,k}{\sqrt{2}\,m}, \frac{(\sqrt{8}+\sqrt{3})\,k}{\sqrt{2}\,m}\right\}.$$

Returning to the eigenvalue equation, the eigenvectors satisfy the equation

$$u_{i1} - \frac{1}{2} u_{i2} = \beta_i \, u_{i1} \qquad \longrightarrow \qquad u_{i2} = 2\,(1-\beta_i)\; u_{i1}$$

so that

$$\omega_1 = \sqrt{\frac{(\sqrt{8}-\sqrt{3})\,k}{\sqrt{2}\,m}}, \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ \sqrt{6}-2 \end{bmatrix},$$
$$\omega_2 = \sqrt{\frac{(\sqrt{8}+\sqrt{3})\,k}{\sqrt{2}\,m}}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -(\sqrt{6}+2) \end{bmatrix}.$$

For each eigenvector the kinetic energy inner product is

$$\boldsymbol{u}_1^T \, \boldsymbol{M} \, \boldsymbol{u}_1 = 14 - 4\sqrt{6}, \qquad \boldsymbol{u}_2^T \, \boldsymbol{M} \, \boldsymbol{u}_2 = 14 + 4\sqrt{6},$$

Finally, the modal equation for the first mode can be written as

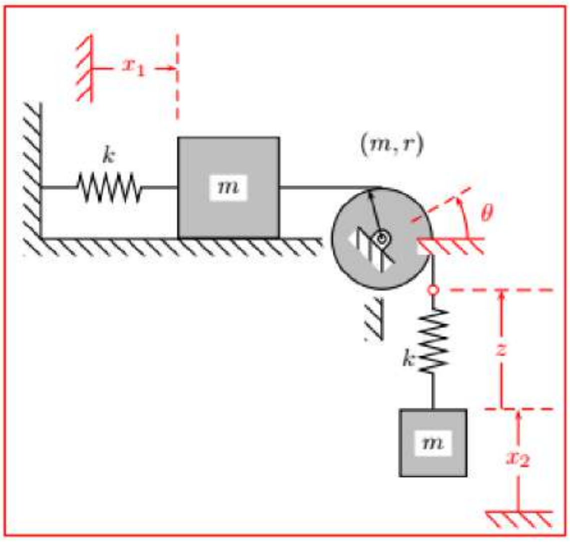
$$\begin{aligned} \left(\boldsymbol{u}_1^T \, \boldsymbol{M} \, \boldsymbol{u}_1\right) \ddot{Q}_1 + \left(\boldsymbol{u}_1^T \, \boldsymbol{K} \, \boldsymbol{u}_1\right) Q_1 &= \boldsymbol{u}_1^T \, \boldsymbol{f}(t), \\ \ddot{Q}_1 + \omega_1^2 \, Q_1 &= \frac{\boldsymbol{u}_1^T \, \boldsymbol{f}(t)}{\boldsymbol{u}_1^T \, \boldsymbol{M} \, \boldsymbol{u}_1}, \\ \ddot{Q}_1 + \left(\frac{(\sqrt{8}-\sqrt{3})\,k}{\sqrt{2}\,m}\right) Q_1 &= \frac{f(t)}{14-4\sqrt{6}}, \end{aligned}$$

while the response of the second mode is governed by

$$\begin{aligned} \left(\boldsymbol{u}_2^T \, \boldsymbol{M} \, \boldsymbol{u}_2\right) \ddot{Q}_2 + \left(\boldsymbol{u}_2^T \, \boldsymbol{K} \, \boldsymbol{u}_2\right) Q_2 &= \boldsymbol{u}_2^T \, \boldsymbol{f}(t), \\ \ddot{Q}_2 + \omega_2^2 \, Q_2 &= \frac{\boldsymbol{u}_2^T \, \boldsymbol{f}(t)}{\boldsymbol{u}_2^T \, \boldsymbol{M} \, \boldsymbol{u}_2}, \\ \ddot{Q}_2 + \left(\frac{(\sqrt{8}+\sqrt{3})\,k}{\sqrt{2}\,m}\right) Q_2 &= \frac{f(t)}{14+4\sqrt{6}}, \end{aligned}$$

Problem: 07

For the system shown in the figure, the surface is assumed to be frictionless. If each block is displaced by a distance d (down and to the right), find the resulting motion of the system.



Solution

We define the coordinates x_1, x_2, θ, and z as shown in the figure, so that

$$x_1 = -r\,\theta, \qquad z = r\,\theta - x_2.$$

With these coordinates, the kinetic and potential energies can be written as

$$\mathcal{T} = \frac{1}{2}\,(m)\,\dot{x}_1^2 + \frac{1}{2}\left(\frac{m\,r^2}{2}\right)\dot{\theta}^2 + \frac{1}{2}\,(m)\,\dot{x}_2^2,$$
$$\mathcal{V} = \frac{1}{2}\,(k)\,x_1^2 + \frac{1}{2}\,(k)\,z^2.$$

Expressing these only in terms of the coordinates x_1 and x_2, we obtain

$$\mathcal{T} = \frac{1}{2}\left(\frac{3\,m}{2}\right)\dot{x}_1^2 + \frac{1}{2}\,(m)\,\dot{x}_2^2,$$
$$\mathcal{V} = \frac{1}{2}\,(k)\,x_1^2 + \frac{1}{2}\,(k)\,(-x_1 - x_2)^2 = \frac{1}{2}\,(2\,k)\,x_1^2 + \frac{1}{2}\,(2\,k)\,x_1\,x_2 + \frac{1}{2}\,(k)\,x_2^2.$$

Therefore, the mass and stiffness matrix can be identifed as

$$\boldsymbol{M} = m\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{K} = k\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{with} \quad \boldsymbol{q}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

and the equations of motion are:

$$m\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + k\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{0}.$$

The solution to this equation requires the solution of an eigenvalue problem of the form

$$(\boldsymbol{M}^{-1}\,\boldsymbol{K})\,\boldsymbol{u} = \frac{k}{m}\begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ 1 & 1 \end{bmatrix}\boldsymbol{u} = \lambda\,\boldsymbol{u},$$

which is determined from the characteristic equation

$$\det\left(\boldsymbol{M}^{-1}\,\boldsymbol{K} - \lambda\,\boldsymbol{I}\right) = \frac{k}{m}\left[\left(\frac{4}{3} - \beta\right)(1 - \beta) - \frac{2}{3}\right] = 0,$$

with $\lambda = \frac{k}{m}\,\beta$. This quadratic equation has the solution

$$\beta = \frac{7 \pm 5}{6} = \left\{\frac{1}{3},\,2\right\} \quad \longrightarrow \quad \omega = \left\{\sqrt{\frac{k}{3\,m}},\,\sqrt{\frac{2\,k}{m}}\right\}$$

With this, the eigenvectors are determined by returning to the original eigenvalue equation

$$\begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix} = \beta_i\begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix}, \quad \longrightarrow \quad u_{i2} = \frac{3\,\beta_i - 4}{2}\,u_{i1}$$

In addition, normalizing the eigenvectors by the mass matrix

$$1 = \boldsymbol{u}_i^T \boldsymbol{M} \boldsymbol{u}_i = \begin{bmatrix} u_{i1} & u_{i2} \end{bmatrix} \begin{bmatrix} \frac{3m}{2} & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix} = m\, u_{i1}^2 \left(\frac{6 + (3\,\beta_i - 4)^2}{4} \right).$$

Solving for u_{i1} the normalized eigenvectors are

$$\boldsymbol{u}_1 = \begin{bmatrix} \frac{2}{\sqrt{15\,m}} \\ -\frac{3}{\sqrt{15\,m}} \end{bmatrix}, \qquad \boldsymbol{u}_2 = \begin{bmatrix} \frac{2}{\sqrt{10\,m}} \\ \frac{2}{\sqrt{10\,m}} \end{bmatrix}.$$

One can easily verify that both $\boldsymbol{u}_1^T \boldsymbol{M} \boldsymbol{u}_2 = 0$ and $\boldsymbol{u}_2^T \boldsymbol{M} \boldsymbol{u}_1 = 0$.

The general solution is written as

$$\boldsymbol{q}(t) = \sum_{i=1}^{2} \Big(A_i \sin(\omega_i\, t) + B_i \cos(\omega_i\, t) \Big)\ \boldsymbol{u}_i$$

subject to the initial conditions

$$\boldsymbol{q}(0) = \begin{bmatrix} d \\ -d \end{bmatrix}, \qquad \dot{\boldsymbol{q}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Premultiplying by $\boldsymbol{u}_i^T \boldsymbol{M}$ yields

$$\boldsymbol{u}_i^T \boldsymbol{M}\, \boldsymbol{q}(0) = \left(\boldsymbol{u}_i^T \boldsymbol{M}\, \boldsymbol{u}_i\right)\ B_i, \qquad \boldsymbol{u}_i^T \boldsymbol{M}\, \dot{\boldsymbol{q}}(0) = \left(\boldsymbol{u}_i^T \boldsymbol{M}\, \boldsymbol{u}_i\right)\ (A_i\, \omega_i).$$

Since $\boldsymbol{u}_i^T \boldsymbol{M} \boldsymbol{u}_i = 1$ from our normalization, the constants are directly solved to be

$$A_1 = 0, \quad B_1 = 6\,d\,\sqrt{\frac{m}{15}}, \qquad A_2 = 0, \quad B_2 = d\,\sqrt{\frac{m}{10}}$$

Finally, the solution to the specific initial conditions becomes

$$\boldsymbol{q}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{d}{5} \left\{ \cos\left(\sqrt{\frac{k}{3\,m}}\, t \right) \begin{bmatrix} 4 \\ -6 \end{bmatrix} + \cos\left(\sqrt{\frac{2\,k}{m}}\, t \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

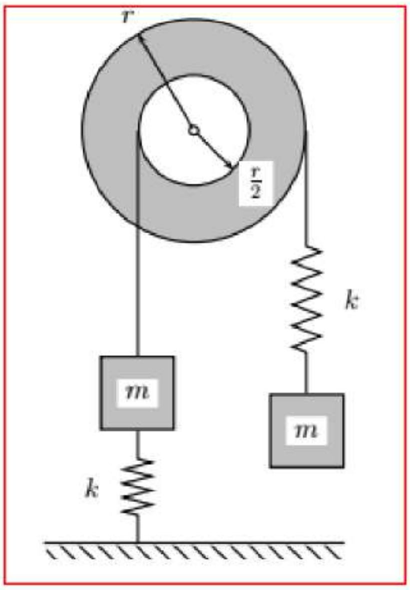
Problem: 08

In the system shown to the right, the pulley has mass m and radius r, so that the moment of inertia about the mass center is $I_G=mr^2/2$

a) What is the degree-of-freedom for this system?

b) Find the governing equations of motion;

c) If $m = 1$ kg and $k = 4$ N/m, what are the frequencies of oscillation for the motion and the corresponding mode shapes, normalized by the kinetic energy inner product?



Solution

a) Let the displacement of the left block, disk, and right block be described as $(-x_1\ \hat{\jmath})$, $(\theta\ \hat{k})$, and $(x_2\ \hat{\jmath})$ respectively. Although x_1 and θ are related by the following constraint:

$$x_1 = \frac{r}{2}\theta,$$

x_2 is independent from the above two coordinates. Therefore, the system has two degrees-of-freedom.

b) With the above coordinates, the kinetic and potential energies can be written as:

$$\begin{aligned}\mathcal{T} &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}\frac{mr^2}{2}\dot{\theta}^2,\\ \mathcal{V} &= \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - r\theta)^2,\end{aligned}$$

Thus, using the above kinematic constraint to eliminate θ, the Lagrangian becomes:

$$\begin{aligned}\mathcal{L} &= \mathcal{T} - \mathcal{V},\\ &= \frac{1}{2}m\left[3\dot{x}_1^2 + \dot{x}_2^2\right] - \frac{1}{2}k\left[5x_1^2 - 4x_1x_2 + x_2^2\right].\end{aligned}$$

Using Lagrange's equations of motion, the governing equations are:

$$\begin{aligned}3m\ \ddot{x}_1 + 5k\ x_1 - 2k\ x_2 &= 0,\\ m\ \ddot{x}_2 - 2k\ x_1 + k\ x_2 &= 0.\end{aligned}$$

c) From the above equations, the mass and stiffness matrices can be written as:

$$\boldsymbol{M} = m \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{K} = k \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

The characteristic matrix, $\boldsymbol{A} = \boldsymbol{M}^{-1}\boldsymbol{K}$ becomes:

$$\boldsymbol{A} = \frac{k}{m} \begin{bmatrix} \frac{5}{3} & -\frac{2}{3} \\ -2 & 1 \end{bmatrix},$$

and the characteristic equation can be written as:

$$\beta^2 - \frac{8}{3}\,\beta + \frac{1}{3} = 0,$$

where, if β is a solution to this equation, $\lambda = \frac{k}{m}\beta$ is an eigenvalue of the characteristic matrix $\boldsymbol{A}$. This quadratic equations has solutions of the from:

$$\beta = \frac{4 \pm \sqrt{13}}{3}.$$

To determine the eigenvectors, we return to the characteristic matrix $\boldsymbol{A}$, so that $\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$. The elements of $\boldsymbol{u}$ then satisfy the equation:

$$\frac{5}{3}\,u_1 - \frac{2}{3}\,u_2 = \beta\,u_1.$$

Thus, if $u_1 = 1$, this yields:

$$u_2 = \frac{1 \mp \sqrt{13}}{2} \quad \text{for} \quad \beta = \frac{4 \pm \sqrt{13}}{3}.$$

Normalizing by the kinetic energy inner product, we find that:

$$\hat{\boldsymbol{u}} = \frac{\boldsymbol{u}}{\sqrt{(\boldsymbol{u}, \boldsymbol{u})_{\boldsymbol{M}}}} = \frac{\boldsymbol{u}}{\sqrt{3u_1^2 + u_2^2}}$$

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